Chapter 2
Differentiation: Basic Concepts

2.1 The Derivative

2. \( f(x) = -3 \)
The difference quotient is
\[
\frac{f(x+h) - f(x)}{h} = \frac{-3 - (-3)}{h} = 0
\]
Then \( f'(x) = \lim_{h \to 0} 0 = 0 \).
The slope of the line tangent to the graph of \( f \) at \( x = 1 \) is \( f'(1) = 0 \).

4. \( f(x) = 2 - 7x \)
The difference quotient is
\[
\frac{f(x+h) - f(x)}{h} = \frac{(2 - 7(x+h)) - (2 - 7x)}{h} = \frac{2 - 7x - 7h - 2 + 7x}{h} = \frac{-7h}{h} = -7
\]
Then \( f'(x) = \lim_{h \to 0} (-7) = -7 \).
The slope of the line tangent to the graph of \( f \) at \( x = -1 \) is \( f'(-1) = -7 \).

6. \( f(x) = x^2 - 1 \)
The difference quotient is
\[
\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - 1 - (x^2 - 1)}{h} = \frac{x^2 + 2hx + h^2 - 1 - x^2 + 1}{h} = \frac{2hx + h^2}{h} = 2x + h
\]
Then \( f'(x) = \lim_{h \to 0} (2x + h) = 2x \).
The slope of the line tangent to the graph of \( f \) at \( x = -1 \) is \( f'(-1) = -2 \).

8. \( f(x) = -x^3 \)
The difference quotient is
\[
\frac{f(x+h) - f(x)}{h} = \frac{(-(x+h)^3) - (-x^3)}{h} = \frac{-x^3 - 3x^2h - 3xh^2 - h^3 + x^3}{h} = \frac{-3x^2h - 3xh^2 - h^3}{h}
\]
Then
\[
\lim_{h \to 0} (-3x^2 - 3xh - h^2) = -3x^2.
\]
The slope of the line tangent to the graph of \( f \) at \( x = 1 \) is \( f'(1) = -3 \).

10. \( f(x) = \frac{1}{x^2} \)
The difference quotient is
\[
\frac{f(x+h) - f(x)}{h} = \frac{1}{(x+h)^2} - \frac{1}{x^2} = \frac{h}{x^2(x^2 + 2hx + h^2)} = \frac{2hx + h^2}{x^2}
\]
Then
\[
\lim_{h \to 0} \frac{-2x - h}{(x^2 + 2hx + h^2)x^2} = -\frac{2}{x^3}.
\]
The slope of the line tangent to the graph of \( f \) at \( x = 2 \) is \( f'(2) = -\frac{1}{4} \).

12. \( f(x) = \sqrt{x} \)
   The difference quotient is
   \[
   \frac{f(x + h) - f(x)}{h} = \frac{\sqrt{x + h} - \sqrt{x}}{h}
   = \frac{\sqrt{x + h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}}
   = \frac{x + h - x}{h(\sqrt{x + h} + \sqrt{x})}
   = \frac{1}{\sqrt{x + h} + \sqrt{x}}
   \]
   Then \( f'(x) = \lim_{h \to 0} \frac{1}{\sqrt{x + h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \)
   The slope of the line tangent to the graph of \( f \) at \( x = 9 \) is \( f'(9) = \frac{1}{6} \).

14. For \( f(x) = 3 \),
   \[
   f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{3 - 3}{h} = 0
   \]
   for all \( x \). So at the point \( c = -4 \), the slope of the tangent line is \( m = f'(-4) = 0 \). The point \((-4,3)\) is on the tangent line so by the point-slope formula the equation of the tangent line is \( y - 3 = 0[x - (-4)] \) or \( y = 3 \).

16. For \( f(x) = 3x \),
   \[
   f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{3x + 3h - 3x}{h}
   = \lim_{h \to 0} \frac{3h}{h}
   = 3
   \]
   for all \( x \). So at the point \( c = 1 \), the slope of the tangent line is \( m = f'(1) = 3 \). The point \((1, 3)\) is on the tangent line so by the point-slope formula the equation of the tangent line is \( y - 3 = 3(x - 1) \) or \( y = 3x \).

18. For \( f(x) = 2 - 3x^2 \),
   \[
   f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(2 - 3(x + h)^2) - (2 - 3x^2)}{h}
   = \lim_{h \to 0} \frac{-6x - 3h}{h}
   = -6x
   \]
   for all \( x \). At the point \( c = 1 \), the slope of the tangent line is \( m = f'(1) = -6 \). The point \((1, -1)\) is on the tangent line so by the point-slope formula the equation of the tangent line is \( y - (-1) = -6(x - 1) \) or \( y = -6x + 5 \).

20. For \( f(x) = \frac{3}{x} \),
   \[
   f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{3}{x + h} - \frac{3}{x}}{h}
   = \lim_{h \to 0} \frac{3(x - (x + h))}{hx(x + h)}
   = \lim_{h \to 0} \frac{-6x - 3h}{hx^2}
   = \frac{-6}{x^3}
   \]
   At the point \( c = \frac{1}{2} \), the slope of the tangent line is \( m = f'(\frac{1}{2}) = -48 \). The point \((\frac{1}{2}, 12)\) is on the tangent line so by the point-slope formula the equation of the tangent line is \( y - 12 = -48 \left(x - \frac{1}{2}\right) \) or \( y = -48x + 36 \).
22. For \( f(x) = \frac{1}{\sqrt{x}} \),
\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
= \lim_{h \to 0} \frac{1/\sqrt{x+h} - 1/\sqrt{x}}{h}
= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x^2 + xh}}
= \lim_{h \to 0} \frac{-1}{h\sqrt{x^2 + xh}(\sqrt{x} + \sqrt{x+h})}
= \frac{-1}{2\sqrt{x^2}}
\]
So at the point \( c = 1 \), the slope of the tangent line is \( f'(1) = -\frac{1}{2} \). The point \((1, 1)\) is on the tangent line so by the point-slope formula, the equation of the tangent line is \( y - 1 = -\frac{1}{2}(x-1) \) or \( y = -\frac{1}{2}x + \frac{3}{2} \).

24. From Exercise 7 of this section \( f'(x) = 3x^2 \). At the point \( c = 1 \), the slope of the tangent line is \( m = f'(1) = 3 \). The point \((1,0)\) is on the tangent line so by the point-slope formula, the equation of the tangent line is \( y - 0 = 3(x-1) \) or \( y = 3x - 3 \).

26. For \( f(x) = -17 \), \( \frac{dy}{dx} \) at \( x_0 = 14 \) is
\[
f'(14) = \lim_{h \to 0} \frac{f(14+h) - f(14)}{h}
= \lim_{h \to 0} \frac{-17 - (-17)}{h}
= \lim_{h \to 0} \frac{0}{h}
= 0
\]

28. For \( f(x) = 6 - 2x \), \( \frac{dy}{dx} \) at \( x_0 = 3 \) is
\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
= \lim_{h \to 0} \frac{(6 - 2(x+h)) - (6 - 2x)}{h}
= \lim_{h \to 0} \frac{-2h}{h}
= -2
\]

30. For \( f(x) = x^2 - 2x \), \( \frac{dy}{dx} \) at \( x_0 = 1 \) is
\[
f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}
= \lim_{h \to 0} \frac{((1+h)^2 - 2(1+h)) - (1^2 - 2(1))}{h}
= \lim_{h \to 0} \frac{h^2}{h}
= 0
\]

32. For \( f(x) = \frac{1}{2-x} \), \( \frac{dy}{dx} \) at \( x_0 = -3 \) is
\[
f'(-3) = \lim_{h \to 0} \frac{f(-3+h) - f(-3)}{h}
= \lim_{h \to 0} \frac{\frac{1}{2-(-3+h)} - \frac{1}{2-(-3)}}{h}
= \lim_{h \to 0} \frac{1}{5(5-h)}
= \frac{1}{25}
\]

34. (a) \( m = \frac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0} \)
\[
= \frac{\left(2\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2\right) - (2(0) - 0^2)}{\frac{1}{2}}
= \frac{\frac{3}{2} - 0}{\frac{1}{2}}
= \frac{3}{2}
\]
The answer is part (a) is a relatively good approximation to the slope of the tangent line.

36. (a) \[ m = \frac{f\left(\frac{1}{2}\right) - f(-1)}{-\frac{1}{2} - (-1)} = \frac{\frac{1}{2} - \left(-\frac{1}{2}\right)}{1} = \frac{1}{2} \]

(b) \[ f'(x) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{\sqrt{1 + h} - \sqrt{1}}{h} = \lim_{h \to 0} \frac{\sqrt{1 + h} - 1}{h} = \lim_{h \to 0} \frac{1}{\sqrt{1 + h} + 1} = \frac{1}{2} \]

The answer in part (a) is a relatively good approximation to the slope of the tangent line.

38. (a) \[ f_{\text{ave}}(x) = \frac{f\left(\frac{1}{2}\right) - f(0)}{x - 0} = \frac{\frac{1}{2} - 0}{\frac{1}{2}} = 1 \]

(b) \[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{1 + h} - \sqrt{1}}{h} = \lim_{h \to 0} \frac{\sqrt{1 + h} - 1}{h} = \lim_{h \to 0} \frac{1}{\sqrt{1 + h} + 1} = \frac{1}{2} \]

The answer in part (a) is a relatively good approximation to the average rate of change.

40. (a) \[ s_{\text{ave}} = \frac{s\left(\frac{1}{4}\right) - s(1)}{\frac{1}{4} - 1} = \frac{\frac{1}{4} - \sqrt{1}}{-\frac{3}{4}} = \frac{-1}{-\frac{3}{4}} = \frac{2}{3} \]

(b) \[ f'(1) = \lim_{h \to 0} \frac{s(1 + h) - s(1)}{h} = \lim_{h \to 0} \frac{\sqrt{1 + h} - \sqrt{1}}{h} = \lim_{h \to 0} \frac{\sqrt{1 + h} - 1}{h} = \lim_{h \to 0} \frac{1}{\sqrt{1 + h} + 1} = \frac{1}{2} \]

The answer in part (a) is a relatively good approximation to the instantaneous rate of change.

42. (a) ... the average rate of change of revenue when the production level changes from \( x_0 \) to \( x_0 + h \) units. 
... the instantaneous rate of change of revenue when the production level is \( x_0 \) units.
(b) ... the average rate of change in the fuel level, in lb/ft, as the rocket travels between \( x_0 \) and \( x_0 + h \) feet above the ground.

... the instantaneous rate in fuel level when the rocket is \( x_0 \) feet above the ground.

(c) ... the average rate of change in volume of the growth as the drug dosage changes from \( x_0 \) to \( x_0 + h \) mg.

... the instantaneous rate in the growth’s volume when \( x_0 \) mg of the drug have been injected.

44. Answers will vary. Drawing a tangent line at each of the indicated points on the curve shows the population is growing at approximately 10/day after 20 days and 8/day after 36 days. The tangent line slope is steepest between 24 and 30 days at approximately 27 days.

46. (a) Profit = (number sold)(profit on each)

Profit on each

= selling price – cost to obtain

\[ P(p) = (120 - p)(p - 50) \]

Since \( q = 120 - p \), \( p = 120 - q \).

\[ P(q) = q(120 - q) - 50 \]

or \( P(q) = q(70 - q) = 70q - q^2 \).

(b) The average rate as \( q \) increases from \( q = 0 \) to \( q = 20 \) is

\[ \frac{P(20) - P(0)}{20} = \frac{[70(20) - (20)^2] - 0}{20} = \frac{28}{20} = \$50 \text{ per recorder} \]

(c) The rate the profit is changing at \( q = 20 \) is \( P'(20) \).

The difference quotient is

\[ \lim_{h \to 0} \frac{P(q + h) - P(q)}{h} = \frac{70[q + h] - (q + h)^2 - [70q - q^2]}{h} \]

\[ = \frac{70q + 70h - q^2 - 2qh - h^2 - 70q + q^2}{h} \]

\[ = \frac{70h - 2qh - h^2}{h} \]

\[ = 70 - 2q - h \]

\[ P'(q) = 70 - 2q \]

\[ P'(20) = 70 - 2(20) = \$30 \text{ per recorder.} \]

Since \( P'(20) \) is positive, profit is increasing.

48. (a) \( Q_{ave} = \frac{Q(3,100) - Q(3,025)}{3,100 - 3,025} \)

\[ = \frac{3,100 \sqrt{3,100} - 3,100 \sqrt{3,025}}{75} \]

\[ = \frac{3,100(10 \sqrt{31} - 55)}{75} \]

\[ \approx 28.01 \]

The average rate of change in output is about 28 units per worker-hour.
(b) $Q'(3,025) = \lim_{h \to 0} \frac{Q(3,025 + h) - Q(3,025)}{h}$
\begin{align*}
&= \lim_{h \to 0} \frac{3,100\sqrt{3,025 + h} - 3,100\sqrt{3,025}}{h} \\
&= \lim_{h \to 0} \frac{3,100(\sqrt{3,025 + h} - 3,025)}{h} \\
&= \lim_{h \to 0} \frac{3,100(3,025 + h - 3,025)}{h\left(\sqrt{3,025 + h} + 55\right)} \\
&= \lim_{h \to 0} \frac{3,100}{\sqrt{3,025 + h} + 55} \\
&= \frac{3,100}{110} \\
&= 28.2
\end{align*}

The instantaneous rate of change is 28.2 units per worker-hour.

50. (a) $E(x) = x \cdot D(x)$
\begin{align*}
&= x(-35x + 200) \\
&= -35x^2 + 200x
\end{align*}

(b) $E_{\text{ave}} = \frac{E(5) - E(4)}{5 - 4}$
\begin{align*}
&= -35(5)^2 + 200(5) - (-35(4)^2 + 200(4)) \\
&= 125 - 240 \\
&= -115
\end{align*}

The average change in consumer expenditures is $-115 per unit.

(c) $E'(4) = \lim_{h \to 0} \frac{E(4 + h) - E(4)}{h}$
\begin{align*}
&= \lim_{h \to 0} \frac{-35(4 + h)^2 + 200(4 + h) - (-35(4)^2 + 200(4))}{h} \\
&= \lim_{h \to 0} \frac{-35h^2 - 80h}{h} \\
&= \lim_{h \to 0} (-35h - 80) \\
&= -80
\end{align*}

The instantaneous rate of change is $-80 per unit when $x = 4$. The expenditure is decreasing when $x = 4$.

52. (a) If $P(t)$ represents the blood pressure function then $P(0.7) = 80$, $P(0.75) = 77$, and $P(0.8) = 85$. The average rate of change on $[0.7,0.75]$ is approximately $\frac{77 - 80}{0.5} = -6$ mm/sec while on $[0.75, 0.8]$ the average rate of change is about $\frac{85 - 77}{0.5} = 16$ mm/sec. The rate of change is greater in magnitude in the period following the burst of blood.

(b) Writing exercise—answers will vary.
54. (a) The rocket is  
\[ h(40) = -800 + 1200 = 400 \text{ feet above ground.} \]

(b) The average velocity between 0 and 40 seconds is given by  
\[ \frac{h(40) - h(0)}{40} = \frac{400}{40} = 10 \text{ feet/second.} \]

(c) \( h'(0) = 30 \text{ ft/sec} \) and \( h'(40) = -10 \text{ ft/sec} \). The negative sign in the second velocity indicates the rocket is falling.

56. (a)  
\[ f'(x) = \lim_{h \to 0} \frac{(3(x+h)-2)-(3x-2)}{h} \]
\[ = \lim_{h \to 0} \frac{3h}{h} \]
\[ = 3 \]

(b) At \( x = -1, \ y = 3(-1) - 2 = -5 \) and \((-1,-5)\) is a point on the tangent line. Using the point-slope formula with \( m = 3 \) gives \( y - (-5) = 3(x - (-1)) \) or \( y = 3x - 2 \).

(c) The line tangent to a straight line at any point is the line itself.

58. (a) For \( f(x) = x^2 + 3x \), the derivative is  
\[ f'(x) = \lim_{h \to 0} \frac{(x+h)^2 + 3(x+h) - (x^2 + 3x)}{h} \]
\[ = \lim_{h \to 0} \frac{x^2 + 2hx + h^2 + 3x + 3h - x^2 - 3x}{h} \]
\[ = \lim_{h \to 0} (2x + h + 3) = 2x + 3 \]

(b) For \( g(x) = x^2 \), the derivative is  
\[ g'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \]
\[ = \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \]
\[ = \lim_{h \to 0} (2x + h) \]
\[ = 2x \]

While for \( h(x) = 3x \), the derivative is  
\[ h'(x) = \lim_{h \to 0} \frac{3(x+h) - 3x}{h} = \lim_{h \to 0} \frac{3h}{h} = 3 \]

(c) The derivative of the sum is the sum of the derivatives.

(d) The derivative of \( f(x) \) is the sum of the derivative of \( g(x) \) and \( h(x) \).

60. If \( y = mx + b \) then  
\[ \frac{dy}{dx} = \lim_{h \to 0} \frac{m(x+h)+b-(mx+b)}{h} \]
\[ = \lim_{h \to 0} \frac{mh}{h} \]
\[ = \lim_{h \to 0} m \]
\[ = m, \text{ a constant.} \]

62. (a) Write any number \( x \) as \( x = c + h \). If the value of \( x \) is approaching \( c \), then \( h \) is approaching 0 and vice versa. Thus the indicated limit is the same as the limit in the definition of the derivative. Less formally, note that if \( x \neq c \) then \( f(x) - f(c) \) is the slope of a secant line. As \( x \) approaches \( c \) the slopes of the secant lines approach the slope of the tangent at \( c \).

(b)  
\[ \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \]
\[ = \lim_{x \to c} \left[ \frac{f(x) - f(c)}{x - c} \right] \lim (x - c) \]
\[ = \lim_{x \to c} \left[ \frac{f(x) - f(c)}{x - c} \right] \lim (x - c) \]
\[ = f'(c) \cdot 0 \]
\[ = 0 \]
using part (a) for the first limit on the right.

(c) Using the properties of limits and the result of part (b)
0 = \lim_{x \to c} [f(x) - f(c)] \\
= \lim_{x \to c} f(x) - \lim_{x \to c} f(c) \\
= \lim_{x \to c} f(x) - f(c) \\
so \lim_{x \to c} f(x) = f(c) meaning f(x) is continuous at \( x = c \).

64. Using the TRACE feature of a calculator with the graph of \( y = 2x^3 - 0.8x^2 + 4 \) shows a peak at \( x = 0 \) and a valley at \( x = 0.2667 \). Note the peaks and valleys are hard to see on the graph unless a small rectangle such as \([-0.3, 0.5] \times [3.8, 4.1]\) is used.

2.2 Techniques of Differentiation

2. \( y = 3 \) \\
\( \frac{dy}{dx} = 0 \)

4. \( y = -2x + 7 \) \\
\( \frac{dy}{dx} = -2(1) + 0 = -2 \)

6. \( y = x^{7/3} \) \\
\( \frac{dy}{dx} = \frac{7}{3}x^{7/3-1} = \frac{7}{3}x^{4/3} \)

8. \( y = 4 - x^{-1.2} \) \\
\( \frac{dy}{dx} = 0 - (-1.2)x^{-1.2-1} = 1.2x^{-2.2} \)

10. \( y = \frac{4}{3}\pi r^3 \) \\
\( \frac{dy}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2 \)

12. \( y = 2\sqrt[4]{x^3} = 2x^{3/4} \) \\
\( \frac{dy}{dx} = 2\left(\frac{3}{4}\right)x^{3/4-1} = \frac{3}{2}x^{-1/4} = \frac{3}{2\sqrt[4]{x}} \)

14. \( y = \frac{3}{2t^2} = \frac{3}{2}t^{-2} \) \\
\( \frac{dy}{dt} = \frac{3}{2}(-2t^{-3}) = -\frac{3}{t^3} \)

16. \( y = 3x^5 - 4x^3 + 9x - 6 \) \\
\( \frac{dy}{dx} = 3(5x^4) - 4(3x^2) + 9(1) - 0 \) \\
= \( 15x^4 - 12x^2 + 9 \)

18. \( f(x) = \frac{1}{4}x^8 - \frac{1}{2}x^6 - x + 2 \) \\
\( f'(x) = 8 \cdot \frac{1}{4}x^7 - 6 \cdot \frac{1}{2}x^5 - 1 + 0 \) \\
= \( 2x^7 - 3x^5 - 1 \)

20. \( f(u) = 0.07u^4 - 1.2lu^3 + 3u - 5.2 \) \\
\( f'(u) = 4(0.07u^3) - 3(1.2lu^2) + 3 - 0 \) \\
= \( 0.28u^3 - 3.63u^2 + 3 \)

22. \( y = \frac{3}{x} - \frac{2}{x^2} + \frac{2}{3x^3} = 3x^{-1} - 2x^{-2} + \frac{2}{3}x^{-3} \) \\
\( \frac{dy}{dx} = (-1)(3x^{-2}) - (-2)(2x^{-3}) + (-3)\left(\frac{2}{3}x^{-4}\right) \) \\
= \( -3x^{-2} + 4x^{-3} - 2x^{-4} \) \\
= \( -\frac{3}{x^2} + \frac{4}{x^3} - \frac{2}{x^4} \)

24. \( f(t) = 2\sqrt{t^3} + \frac{4}{\sqrt{t}} - \sqrt{2} \) \\
= \( 2t^{3/2} + 4t^{-1/2} - \sqrt{2} \) \\
\( f'(t) = \frac{3}{2}(2t^{3/2-1}) + \frac{-1}{2}(4t^{-1/2-1}) - 0 \) \\
= \( 3t^{1/2} - 2t^{-3/2} \) \\
= \( 3\sqrt{t} - \frac{2}{\sqrt{t^3}} \)

26. \( y = \frac{-7}{x^{12}} + \frac{5}{x^{2.1}} = -7x^{-1.2} + 5x^{2.1} \) \\
\( \frac{dy}{dx} = -1.2(-7x^{-1.2-1}) + 2.1(5x^{2.1-1}) \) \\
= \( 8.4x^{-2.2} + 10.5x^{1.1} \) \\
= \( \frac{8.4}{x^{2.2}} + 10.5x^{1.1} \)

28. \( y = x^2(x^3 - 6x + 7) = x^5 - 6x^3 + 7x^2 \) \\
\( \frac{dy}{dx} = 5x^4 - 18x^2 + 14x \)
30. Given \( y = x^5 - 3x^3 - 5x + 2 \) and the point \((1, -5)\), then \( \frac{dy}{dx} = 5x^4 - 9x^2 - 5 \) and the slope of the tangent line at \( x = 1 \) is 

\[ m = 5(1^4) - 9(1^2) - 5 = -9. \]

The equation of the tangent line is then

\[ y = -9(x - 1) \text{ or } y = -9x + 4. \]

32. Given \( y = \sqrt[3]{x^5} - x^2 + \frac{16}{x^2} \) and the point \((4, 7)\), then \( \frac{dy}{dx} = \frac{3}{2} \sqrt{x} - 2x - \frac{32}{x^3} \) and the slope of the tangent line at \( x = 4 \) is

\[ m = \frac{3}{2} \sqrt{4} - 2(4) - \frac{32}{4^3} = -\frac{11}{2}. \]

The equation of the tangent line is then

\[ y = -\frac{11}{2} (x - 4) \text{ or } y = -\frac{11}{2} x + 15. \]

34. Given \( y = 2x^4 - \sqrt[3]{x} + \frac{3}{x} \) and the point \((1, 4)\), then \( \frac{dy}{dx} = 8x^3 - \frac{1}{2\sqrt{x}} - \frac{3}{x^2} \) and the slope of the tangent line at \( x = 1 \) is

\[ m = 8(1^3) - \frac{1}{2\sqrt{1}} - \frac{3}{1^2} = \frac{9}{2}. \]

The equation of the tangent line is then

\[ y - 4 = \frac{9}{2} (x - 1) \text{ or } y = \frac{9}{2} x - \frac{1}{2}. \]

36. \( f(x) = x^4 - 3x^3 + 2x^2 - 6; \ x = 2 \)

\( f'(x) = 4x^3 - 9x^2 + 4x \)

\( f(2) = 16 - 24 + 8 - 6 = -6 \) so \((2, -6)\) is a point on the tangent line. The slope is

\[ m = f'(2) = 32 - 36 + 8 = 4. \]

The equation of the tangent line is \( y - (-6) = 4(x - 2) \) or \( y = 4x - 14 \).

38. \( f(x) = x^3 + \sqrt{x}; \ x = 4 \)

\[ f'(x) = 3x^2 + \frac{1}{2\sqrt{x}} \]

\( f(4) = 64 + 2 = 66 \) so \((4, 66)\) is a point on the tangent line. The slope is

\[ m = f'(4) = 48 + \frac{1}{4} = \frac{193}{4}. \]

The equation of the tangent line is \( y - 66 = \frac{193}{4}(x - 4) \) or 

\[ y = \frac{193}{4}x - 127. \]

40. \( f(x) = x(\sqrt{x}) - 1 = x^{3/2} - x; \ x = 4 \)

\( f'(x) = \frac{3}{2\sqrt{x}} \)

\( f(4) = 8 - 4 = 4 \) so \((4, 4)\) is a point on the tangent line. The slope is

\[ m = f'(4) = 3 - 1 = 2. \]

The equation of the tangent line is \( y - 4 = 2(x - 4) \) or \( y = 2x - 4 \).

42. \( f(x) = x^3 - 3x + 5; \ x = 2 \)

\[ f'(x) = 3x^2 - 3 \]

\[ f'(2) = 3(4) - 3 = 9 \]

44. \( f(x) = \sqrt{x} + 5x; \ x = 4 \)

\[ f'(x) = \frac{1}{2\sqrt{x}} + 5 \]

\[ f'(4) = \frac{1}{2(2)} + 5 = \frac{21}{4} \]

46. \( f(x) = \frac{2}{x} - x\sqrt{x}; \ x = 1 \)

\[ f'(x) = \frac{-2}{x^2} - \frac{3}{2\sqrt{x}} \]

\[ f'(1) = -2 - \frac{3}{2} = -\frac{7}{2} \]

48. \( f(x) = x + \frac{1}{x} = x + x^{-1}; \ f(1) = 1 + 1 = 2 \)

\[ f'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2}; \ f'(1) = 1 - 1 = 0 \]

At \( c = 1 \), the relative rate of change is

\[ \frac{f'(1)}{f(1)} = \frac{0}{2} = 0 \]

50. \( f(x) = (4 - x)x^{-1} = 4x^{-1} - 1; \)

\[ f'(x) = -4x^{-2} - 1 \]

\[ f(3) = \frac{4}{3} - 1 = \frac{1}{3} \]
\[ f'(x) = -4x^{-2}; \quad f'(3) = -\frac{4}{9} \]

At \( c = 3 \), the relative rate of change is
\[ \frac{f'(3)}{f(3)} = \frac{-\frac{4}{9}}{\frac{8}{3}} = -\frac{4}{9} \cdot \frac{3}{8} = -\frac{1}{6}. \]

52. (a) Since \( f(x) = -x^3 + 6x^2 + 15x \) is the number of radios assembled \( x \) hours after 8:00 A.M., the rate of assembly after \( x \) hours is
\[ f'(x) = -3x^2 + 12x + 15 \text{ radios per hour.} \]

(b) The rate of assembly at 9:00 A.M. \( (x = 1) \) is
\[ f'(1) = -3 + 12 + 15 = 24 \text{ radios per hour.} \]

(c) The actual number of radios assembled between 9:00 A.M. and 10:00 A.M. is
\[ f(2) - f(1) = 46 - 20 = 26 \text{ radios.} \]

54. (a) Since \( N(x) = 6x^3 + 500x + 8,000 \) is the number of people using rapid transit after \( x \) weeks, the rate at which system use is changing after \( x \) weeks is
\[ N'(x) = 18x^2 + 500 \text{ commuters per week.} \]
After 8 weeks this rate is
\[ N'(8) = 18(8^2) + 500 = 1652 \text{ users per week.} \]

(b) The actual change in usage during the 8th week is
\[ N(8) - N(7) = 15,072 - 13,558 = 1,514 \text{ riders.} \]

56. \( M(x) = 2,300 + \frac{125}{x} - \frac{517}{x^2} \)
\[ M'(x) = -\frac{125}{x^2} + \frac{1034}{x^3} \]
\[ M'(9) = -\frac{125}{9^2} + \frac{1034}{9^3} = -0.125 \]
Sales are decreasing at a rate of approximately 1/8 motorcycle per $1,000 of advertising.

58. (a) \( P(t) = t^2 + 200t + 10,000 = (t + 100)^2 \)
\[ P'(t) = 2t + 200 = 2(t + 100) \]
The percentage rate of change is
\[ \frac{P'(t)}{P(t)} = \frac{200(t + 100)}{(t + 100)^2} = \frac{200}{t + 100}. \]

(b) The percentage rate of changes approaches 0 since \( \lim_{t \to \infty} \frac{200}{t + 100} = 0. \)

60. (a) \( N(t) = 5,175 - t^3(t - 8) \)
\[ = 5,175 - t^4 + 8t^3 \]
\[ N'(3) = -4(3^3) + 8 \cdot 3(3^2) = 108 \text{ people per week.} \]

(b) The percentage rate of change of \( N \) is given by
\[ \frac{N'(t)}{N(t)} = \frac{100(-4t^3 + 24t^2)}{5,175 - t^4 + 8t^3}. \]
A graph of this function shows that it never exceeds 25%.

(c) Writing exercise—answers will vary.

62. (a) Since \( C(t) = 100t^2 + 400t + 5,000 \) is the circulation \( t \) years from now, the rate of change of the circulation in \( t \) years is
\[ C'(t) = 200t + 400 \text{ newspapers per year.} \]

(b) The rate of change of the circulation 5 years from now is
\[ C'(5) = 200(5) + 400 = 1,400 \text{ newspapers per year.} \] The circulation is increasing.
(c) The actual change in the circulation during the 6th year is
\[ C(6) - C(5) = 11,000 - 9,500 = 1,500 \text{ newspapers}. \]

64. Let \( G(t) \) be the GDP in billions of dollars where \( t \) is years and \( t = 0 \) represents 1995. Since the GDP is growing at a constant rate, \( G(t) \) is a linear function passing through the points \((0, 125)\) and \((8, 155)\). Then
\[
G(t) = \frac{155 - 125}{8 - 0} t + 125 = \frac{15}{4} t + 125.
\]
In 2010, \( t = 15 \) and the model predicts a GDP of \( G(15) = 181.25 \) billion dollars.

66. \[ P = \frac{4}{3} \pi N \left( \frac{\mu^2}{3kT} \right) = \left( \frac{4\pi N \mu^2}{9k} \right) T^{-1} \]
\[
dP dt = -\left( \frac{4\pi N \mu^2}{9k} \right) T^{-2} = \frac{4\pi N \mu^2}{9kT^2} \]

68. (a) \[ s(t) = t^2 - 2t + 6 \text{ for } 0 \leq t \leq 2 \]
\[ v(t) = 2t - 2 \]
\[ a(t) = 2 \]

(b) The particle is stationary when \( v(t) = 2t - 2 = 0 \) which is at time \( t = 1 \).

70. (a) \[ s(t) = t^3 - 9t^2 + 15t + 25 \text{ for } 0 \leq t \leq 6 \]
\[ v(t) = 3t^2 - 18t + 15 = 3(t - 1)(t - 5) \]
\[ a(t) = 6t - 18 = 6(t - 3) \]

(b) The particle is stationary when \( v(t) = 3(t - 1)(t - 5) = 0 \) which is at times
\[ t = 1 \text{ and } t = 5. \]

72. (a) Since the initial velocity is \( V_0 = 0 \) feet per second, the initial height is \( H_0 = 144 \) feet and \( g = 32 \) feet per second per second, the height of the stone at time \( t \) is
\[
H(t) = -\frac{1}{2} gt^2 + V_0 t + H_0 = -16t^2 + 144. \]
The stone hits the ground when \( H(t) = -16t^2 + 144 = 0 \), that is when \( t^2 = 9 \) or after \( t = 3 \) seconds.

(b) The velocity at time \( t \) is given by \( H(t) = -32t \). When the stone hits the ground, its velocity is \( H'(3) = -96 \) feet per second.

74. Let \( g \) be the acceleration due to gravity for the planet our spy is on. Since he throws the rock from ground level, \( H_0 = 0 \), and the equation describing the rock’s height is
\[
H(t) = -\frac{1}{2} gt^2 + V_0 t
\]
When \( t = 5 \) the height is 0 so
\[
-\frac{25}{2} g + 5V_0 = 0
\]
The rock reaches its maximum height halfway through its trip at \( t = \frac{5}{2} \).
Therefore \( H \left( \frac{5}{2} \right) = \frac{25}{8} g + \frac{5}{2} V_0 = 37.5. \)

76. Let \( (x, y) \) be a point on the curve where the tangent line goes through \((0, 0)\). Then the slope of the tangent line is equal to \( \frac{y - 0}{x - 0} = \frac{y}{x} \). The slope is also given by
\[
f'(x) = 2x - 4. \text{ Thus } \frac{y}{x} = 2x - 4 \text{ or } y = 2x^2 - 4x.
\]
Since \((x, y)\) is a point on the curve, we must have \( y = x^2 - 4x + 25 \). Setting the two expressions for \( y \) equal to each other gives
\[
x^2 - 4x + 25 = 2x^2 - 4x
\]
\[
x^2 = 25 \quad \Rightarrow \quad x = \pm 5
\]
If \( x = -5 \), then \( y = 70 \), the slope is \(-14\) and the tangent line is \( y = -14x \).
If \( x = 5 \), then \( y = 30 \), the slope is 6 and the tangent line is \( y = 6x \).
78. (a) If \( f(x) = x^4 \) then
\[
\begin{align*}
f(x + h) &= (x + h)^4 \\
&= x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 \\
f(x + h) - f(x) &= 4x^3h + 6x^2h^2 + 4xh^3 + h^4 \\
\text{and} \\
\frac{f(x + h) - f(x)}{h} &= 4x^3 + 6x^2h + 4xh^2 + h^3
\end{align*}
\]

(b) If \( f(x) = x^n \) then
\[
\begin{align*}
f(x + h) &= (x + h)^n \\
&= x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + n xh^{n-1} + h^n \\
f(x + h) - f(x) &= nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + nxh^{n-1} + h^n \\
\text{and} \\
\frac{f(x + h) - f(x)}{h} &= nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \ldots + nxh^{n-2} + h^{n-1}
\end{align*}
\]

(c) From part (b)
\[
\frac{f(x + h) - f(x)}{h} = nx^{n-1} + h \left[ \frac{n(n-1)}{2}x^{n-2} + \ldots + h^{n-2} \right]
\]
The first term on the right does not involve \( h \) while the second term approaches 0 as \( h \to 0 \).
Thus \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = nx^{n-1} \).

2.3 Product and Quotient Rules; Higher-Order Derivatives

2. \( f(x) = (x - 5)(1 - 2x) \)
\[
\begin{align*}
f(x) &= (x - 5) \frac{d}{dx} (1 - 2x) + (1 - 2x) \frac{d}{dx} (x - 5) \\
&= -2(x - 5) + 1(1 - 2x) \\
&= 11 - 4x
\end{align*}
\]

4. \( y = 400(15 - x^2)(3x - 2) \)
\[
\begin{align*}
\frac{dy}{dx} &= 400 \frac{d}{dx} [(15 - x^2)(3x - 2)] \\
&= 400 \left[ (15 - x^2) \frac{d}{dx} (3x - 2) + (3x - 2) \frac{d}{dx} (15 - x^2) \right] \\
&= 400 \left[ (15 - x^2)(3) + (3x - 2)(-2x) \right] \\
&= 400(-9x^2 + 4x + 45)
\end{align*}
\]
Chapter 2. Differentiation: Basic Concepts

6. \( f(x) = -3(5x^3 - 2x + 5)(\sqrt{x} + 2x) \)

\[ f'(x) = -3 \left[ (5x^3 - 2x + 5) \left( \frac{1}{2\sqrt{x}} + 2 \right) + (\sqrt{x} + 2x)(15x^2 - 2) \right] \]

\[ = -\frac{105}{2} x^{5/2} - 120x^3 + 9x^{1/2} + 24x - \frac{15}{2}x^{1/2} - 30 \]

8. \( y = \frac{2x - 3}{5x + 4} \)

\[ \frac{dy}{dx} = \frac{(5x + 4) \frac{d}{dx}(2x - 3) - (2x - 3) \frac{d}{dx}(5x + 4)}{(5x + 4)^2} \]

\[ = \frac{2(5x + 4) - 5(2x - 3)}{(5x + 4)^2} \]

\[ = \frac{23}{(5x + 4)^2} \]

10. \( f(x) = \frac{1}{x - 2} \)

\[ f'(x) = \frac{(x - 2)(0) - 1(1)}{(x - 2)^2} = -\frac{1}{(x - 2)^2} \]

12. \( y = \frac{t^2 + 1}{1 - t^2} \)

\[ \frac{dy}{dt} = \frac{(1 - t^2)(2t) - (t^2 + 1)(-2t)}{(1 - t^2)^2} = \frac{4t}{(1 - t^2)^2} \]

14. \( f(t) = \frac{t^2 + 2t + 1}{t^2 + 3t - 1} \)

\[ f'(t) = \frac{(t^2 + 3t - 1)(2t + 2) - (t^2 + 2t + 1)(2t + 3)}{(t^2 + 3t - 1)^2} \]

\[ = \frac{t^2 - 4t - 5}{(t^2 + 3t - 1)^2} \]

16. \( g(x) = \frac{(x^2 + x + 1)(4 - x)}{2x - 1} \)

\[ g'(x) = \frac{\left[ (2x - 1)[-1(x^2 + x + 1) + (4 - x)(2x + 1)] - (x^2 + x + 1)(4 - x)(2) \right]}{(2x - 1)^2} \]

\[ = \frac{-4x^3 + 9x^2 - 6x - 11}{(2x - 1)^2} \]
18. \( f(x) = \left( x + \frac{1}{x} \right)^2 = x^2 + 2 + \frac{1}{x^2} \)
\( f'(x) = 2x - \frac{2}{x^3} \)

20. \( h(x) = \frac{x}{x^2 - 1} + \frac{4 - x}{x^2 + 1} \)
\( h'(x) = \frac{(x^2 - 1)(1) - x(2x)}{(x^2 - 1)^2} + \frac{(x^2 + 1)(-1) - (4 - x)(2x)}{(x^2 + 1)^2} \)
\( = \frac{-x^2 - 1 + x^2 - 8x - 1}{(x^2 - 1)^2} + \frac{-x^2 + 8x - 1}{(x^2 + 1)^2} \)

22. \( y = (x^2 + 3x - 1)(2 - x) \)
\( y' = (x^2 + 3x - 1)(-1) + (2x + 3)(2 - x) \)
At \( x_0 = 1 \), \( y = (3)(1) = 3 \) and \( y' = (3)(-1) + (5)(1) = 2 \). The equation of the tangent line is then \( y - 3 = 2(x - 1) \) or \( y = 2x + 1 \).

24. \( y = \frac{x + 7}{5 - 2x} \)
\( y' = \frac{(5 - 2x)(1) - (x + 7)(-2)}{(5 - 2x)^2} \)
At \( x_0 = 0 \), \( y = \frac{7}{5} \) and \( y' = \frac{5 + 14}{5^2} = \frac{19}{25} \).
The equation of the tangent line is then \( y - \frac{7}{5} = \frac{19}{25}(x - 0) \) or \( y = \frac{19}{25}x + \frac{7}{5} \).

26. \( y = \frac{2x - 1}{1 - x^3} \)
\( y' = \frac{(1 - x^3)(2) - (2x - 1)(-3x^2)}{(1 - x^3)^2} \)
At \( x_0 = 0 \), \( y = -1 \) and \( y' = \frac{2 - 0}{1} = 2 \).
The equation of the tangent line is then \( y - (-1) = 2(x - 0) \) or \( y = 2x - 1 \).

28. \( f(x) = (x - 1)(x^2 - 8x + 7) \)
\( f'(x) = \frac{d}{dx}[(x - 1)(x^2 - 8x + 7)] = \frac{d}{dx}(x^3 - 8x^2 + 7x - 7) \)
\( = 3x^2 - 18x + 15 \)
\( = 3(x^2 - 8x + 5) \)
\( = 3(x - 1)(x - 5) \)
\( f'(x) = 0 \) when \( x = 1 \) and \( x = 5 \).
\( f(1) = (1 - 1)(1^2 - 8 \cdot 1 + 7) = 0 \)
\( f(5) = (5 - 1)(5^2 - 8 \cdot 5 + 7) = -32 \)
The tangent lines at \((1, 0)\) and \((5, -32)\) are horizontal.
30. \[ f(x) = \frac{x^2 + x - 1}{x^2 - x + 1} \]
   \[ f'(x) = \frac{(2x + 1)(x^2 - x + 1) - (x^2 + x - 1)(2x - 1)}{(x^2 - x + 1)^2} \]
   \[= \frac{-2x^2 + 4x}{(x^2 - x + 1)^2} = \frac{-2x(x - 2)}{(x^2 - x + 1)^2} \]
   \[ f'(x) = 0 \text{ when } x = 0 \text{ and } x = 2 \]
   \[ f(0) = \frac{0^2 + 0 - 1}{0^2 - 0 + 1} = -1 \]
   \[ f(2) = \frac{2^2 + 2 - 1}{2^2 - 2 + 1} = \frac{5}{3} \]
   The tangent lines at \((0, -1)\) and \((2, \frac{5}{3})\) are horizontal.

32. \[ y = (x^2 + 2)(x + \sqrt{x}) \]
   \[ \frac{dy}{dx} = (x^2 + 2) \left( 1 + \frac{1}{2\sqrt{x}} \right) + 2x(x + \sqrt{x}) \]
   \[ \text{At } x_0 = 4, \]
   \[ \frac{dy}{dx} = (18 \left( 1 + \frac{1}{4} \right) + 8(6) = 70.5 \]

34. \[ y = \frac{2x - 1}{3x + 5} \]
   \[ \frac{dy}{dx} = \frac{2(3x + 5) - 3(2x - 1)}{(3x + 5)^2} = \frac{13}{(3x + 5)^2} \]
   \[ \text{At } x_0 = 1, \quad \frac{dy}{dx} = \frac{13}{8^2} = \frac{13}{64} \]

36. \[ y = x^2 + 3x - 5 \]
   \[ y' = 2x + 3 \]
   \[ \text{At } x = 0, \ y' = 3 \text{ so the slope of the perpendicular line is } m = -\frac{1}{3}. \]
   The perpendicular line passes through the point \((0, -5)\) and so has equation \[ y = -\frac{1}{3}x - 5. \]

38. \[ y = (x + 3) \left( 1 - \sqrt{x} \right) \]
   \[ y' = (x + 3) \left( -\frac{1}{2\sqrt{x}} \right) + (1 - \sqrt{x}) \]
   At \( x = 1, \ y' = -2 \) so the slope of the perpendicular line is \( m = \frac{1}{2}. \)
   The perpendicular line passes through the point \((1, 0)\) and so has equation \[ y = \frac{1}{2}x - \frac{1}{2}. \]

40. (a) \[ y = 2x^2 - 5x - 3 \]
   \[ y' = 4x - 5 \]
   (b) \[ y = (2x + 1)(x - 3) \]
   \[ y' = (2x + 1)(1) + (2)(x - 3) = 4x - 5 \]

42. \[ f(x) = 5x^{10} - 6x^5 - 27x + 4 \]
   \[ f'(x) = 50x^9 - 30x^4 - 27 \]
   \[ f''(x) = 450x^8 - 120x^3 \]

44. \[ y = 5\sqrt{x} + \frac{3}{x^2} + \frac{1}{3\sqrt{x}} + \frac{1}{2} \]
   \[ \frac{dy}{dx} = \frac{5}{2} x^{-1/2} - 6x^{-3} - \frac{1}{6} x^{-3/2} \]
   \[ \frac{d^2y}{dx^2} = -\frac{5}{4} x^{-3/2} + 18x^{-4} + \frac{1}{4} x^{-5/2} \]
   \[ = -\frac{5}{4x^{3/2}} + \frac{18}{x^4} + \frac{1}{4x^{5/2}} \]

46. \[ y = (x^2 - x) \left( \frac{2x - 1}{x} \right) \]
   \[ \frac{dy}{dx} = (x^2 - x) \left( 2 + \frac{1}{x^2} \right) + (2x - 1) \left( 2x - \frac{1}{x} \right) \]
   \[ = 6x^2 - 4x - 1 \]
   \[ \frac{d^2y}{dx^2} = 12x - 4 \]
48. (a) \[ p(x) = \frac{1,000}{0.3x^2 + 8} \]
\[ p'(x) = \frac{(0.3x^2 + 8)(0) - 1,000(0.6x)}{(0.3x^2 + 8)^2} \]
\[ = \frac{-600x}{(0.3x^2 + 8)^2} \]
when the level of production is 3,000 \((x = 3)\) calculators, demand is changing at the rate of \(p'(3) = -15.72\) dollars per thousand calculators.

(b) \(R(x) = xp(x)\)
\[ R'(x) = xp'(x) + p(x)(1) \]
\[ = x \left( \frac{-600x}{(0.3x^2 + 8)^2} \right) + \frac{1,000}{0.3x^2 + 8} \]
\[ = \frac{-300x^2 + 8,000}{(0.3x^2 + 8)^2} \]
\[ R'(3) = 46.29 \] so revenue is increasing at the rate of $46.29 per thousand calculators.

50. (a) Since profit equals revenue minus cost and revenue equals price times the quantity sold, the profit function \(P(p)\) is given by
\[ P(p) = pR(p) - C(p) \]
\[ = \frac{500p}{p + 3} - (0.2p^2 + 3p + 200) \]

(b) \[ P'(p) = \frac{(p + 3)500 - 500p(1)}{(p + 3)^2} - 0.4p - 3 \]
\[ = \frac{1500}{(p + 3)^2} - 0.4p - 3 \]
When the price is $12 per bottle, \(P'(12) = -1.133\). The profit is decreasing.

52. (a) \[ P(t) = \frac{24t + 10}{t^2 + 1} \]
\[ P'(t) = \frac{(t^2 + 1)(24) - (24t + 10)(2t)}{(t^2 + 1)^2} \]
\[ = \frac{-24t^2 - 20t + 24}{(t^2 + 1)^2} \]
\[ = \frac{-4(2t + 3)(3t - 2)}{(t^2 + 1)^2} \]
\[ P'(1) = -5 \] so the population is decreasing at \(t = 1\).

(b) The population rate of change is 0 at \(t = \frac{2}{3}\), positive for \(t < \frac{2}{3}\) and negative for \(t > \frac{2}{3}\). The population begins to decline after \(t = \frac{2}{3}\) or 40 minutes after the introduction of the toxin.
54. (a) \( C(t) = \frac{2t}{3t^2 + 16} \)

\[ R(t) = C'(t) = \frac{(3t^2 + 16)2 - 2t(6t)}{(3t^2 + 16)^2} = \frac{32 - 6t^2}{(3t^2 + 16)^2} \]

\( R(t) \) is changing at the rate

\[ R'(t) = \frac{(3t^2 + 16)^2(-12t) - (32 - 6t^2)(2)(3t^2 + 16)(6t)}{(3t^2 + 16)^4} = \frac{36t(t^2 - 16)}{(3t^2 + 16)^3} \]

(b) \( C'(1) = \frac{26}{361} \), the concentration is increasing at this time.

(c) \( R(t) \) is positive and the concentration is increasing until \( R(t) = 0 \) or when \( 32 - 6t^2 = 0 \). This occurs when \( t = \frac{4}{\sqrt{3}} \approx 2.3 \) hours (ignoring the negative solution.) The concentration begins to decline after roughly 2.3 hours.

(d) The concentration is changing at a declining rate when \( R'(t) = \frac{36t(t^2 - 16)}{(3t^2 + 16)^3} < 0 \)

or when \( 36t(t^2 - 16) < 0 \) (assuming \( t > 0 \)). This occurs when \( 0 < t < 4 \).

56. (a) \( P(t) = 20 - \frac{6}{t+1} \)

\[ P'(t) = \frac{6}{(t+1)^2} \text{ thousand per year.} \]

(b) \( P'(1) = \frac{3}{2} \) so the rate of change in 1 year will be 1,500 people per year.

(c) During the second year, the population will increase by \( P(2) - P(1) = 1 \) thousand people.

(d) \( P'(9) = 0.06 \) or 60 people per year.

(e) \[ \lim_{t \to \infty} P'(t) = \lim_{t \to \infty} \frac{6}{(t+1)^2} = 0 \] so the rate of population growth approaches 0.

58. (a) \( s(t) = 2t^4 - 5t^3 + t - 3 \)

\( v(t) = s'(t) = 8t^3 - 15t^2 + 1 \)

\( a(t) = v'(t) = s''(t) = 24t^2 - 30t 

= 6t(4t - 5) \)

(b) \( a(t) = 0 \) at \( t = 0 \) and \( t = \frac{5}{4} \).
60. (a) \( s(t) = 4t^{5/2} - 15t^2 + t - 3 \)
\[ v(t) = s'(t) = 10t^{3/2} - 30t + 1 \]
\[ a(t) = v'(t) = s''(t) = 15t^{1/2} - 30 \]
(b) \( a(t) = 0 \) when \( t = 4 \).

62. (a) \( D(t) = 64t + \frac{10}{3} t^2 - \frac{2}{9} t^3 \)
\[ v(t) = D'(t) = 64 + \frac{20}{3} t - \frac{2}{3} t^2 \]
\[ a(t) = v'(t) = D''(t) = \frac{20}{3} - \frac{4}{3} t \]
(b) \( a(6) = \frac{20}{3} - \frac{24}{3} = -4 \) indicating the velocity is decreasing at a rate of approximately 1.33 kilometers per hour.
(c) During the seventh hour, the velocity changes by \( v(7) - v(6) = 78 - 80 = -2 \) km/hr.

64. (a) \( p(x) = \frac{Ax}{B + x^m} \)
\[ p'(x) = \frac{(B + x^m)(A) - (Ax)(mx^{m-1})}{(B + x^m)^2} \]
\[ = \frac{A(B + (1-m)x^m)}{(B + x^m)^2} \]
(b) \( p''(x) = \frac{A(m-1)x^{m-1}[B(1+m)]}{(B + x^m)^3} \)
\[ p''(x) = 0 \text{ when } x = m \sqrt[1-m]{\frac{B(1+m)}{1}} \]

66. \( f(x) = x^5 - 2x^4 + x^3 - 3x^2 + 5x - 6 \)
\[ f'(x) = 5x^4 - 8x^3 + 3x^2 - 6x + 5 \]
\[ f''(x) = 20x^3 - 24x^2 + 6x - 6 \]
\[ f'''(x) = 60x^2 - 48x + 6 \]
\[ f^{(4)}(x) = 120x - 48 \]
68. (a) \[ \frac{d}{dx}[fgh] = \frac{d}{dx}[(fg)h] \]
\[ = (fg) \frac{dh}{dx} + h \frac{d}{dx} (fg) \]
\[ = fg \left( \frac{dh}{dx} + h \left( f \frac{dg}{dx} + g \frac{df}{dx} \right) \right) \]
\[ = fg \frac{dh}{dx} + fh \frac{dg}{dx} + gh \frac{df}{dx} \]

(b) \( y = (2x + 1)(x - 3)(1 - 4x) \)
\( y' = (2x + 1)(x - 3)(-4) + (2x + 1)(1)(1 - 4x) + (2)(x - 3)(1 - 4x) \)
\[ = -24x^2 + 44x + 7 \]

70. \[ \frac{d}{dx}[cf] = c \frac{df}{dx} + f \frac{d}{dx} c \]
\[ = c \frac{df}{dx} + f(0) \]
\[ = c \frac{df}{dx} \]

72. Suppose \( n \) is a negative integer so that \( n = -p \) where \( p \) is a positive integer. Then \( \frac{d}{dx} x^p = px^{p-1} \)
since the power rule applies to positive integer powers. Now note
\[ \frac{d}{dx}[x^n] = \frac{d}{dx}[x^{-p}] = \frac{d}{dx} \left( \frac{1}{x^p} \right) \]
\[ = \frac{x^p(0) - 1(px^{p-1})}{x^{2p}} \]
\[ = -p(x^{p-1})(x^{-2p}) \]
\[ = -px^{-p-1} = nx^{n-1} \]
proving the power rule for negative integer powers.

74. 
\( f'(x) = 0 \) when \( x = 0.633 \) and \( x = -2.633 \).
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76. \( f(x) = x^3(x - 2)^2 \)
\[
f'(x) = x^3(2(x - 2)) + 3x^2(x - 2)^2
= x^2(5x - 6)(x - 2)
\]
The \( x \) intercepts of the graph of \( f'(x) \)
occurs at \( x = 0, x = \frac{6}{5}, \) and \( x = 2 \). The
function \( f(x) \) has a maximum at \( x = \frac{6}{5} \)
and a minimum at \( x = 2 \). The maximum and
minimum of \( f(x) \) correspond to points
where the tangent line is horizontal, that
is, where \( f'(x) = 0 \).

2.4 The Chain Rule

2. \( y = 1 - 3u^2; \quad u = 3 - 2x \)
\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
= (-6u)(-2)
= 12(3 - 2x)
= -24x + 36
\]

4. \( y = 2u^2 - u + 5; \quad u = 1 - x^2 \)
\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
= (4u - 1)(-2x^2)
= (4 - 4x^2 - 1)(-2x)
= -2x(-4x^2 + 3)
= 8x^3 - 6x
\]

6. \( y = \frac{1}{u}; \quad u = 3x^2 + 5 \)
\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
= -\left(\frac{6x}{u^2}\right)
= -\frac{6x}{(3x^2 + 5)^2}
\]

8. \( y = \frac{1}{\sqrt{u}}; \quad u = x^2 - 9 \)
\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
= -\left(\frac{1}{2u^{3/2}}\right)(2x)
= -\frac{x}{(x^2 - 9)^{3/2}}
\]

10. \( y = u^3 + u; \quad u = \frac{1}{\sqrt{x}} \)
\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
= (3u^2 + 1)\left(-\frac{1}{2x^{3/2}}\right)
= \left(3 + \frac{1}{x}\right)\left(-\frac{1}{2x^{3/2}}\right)
= -\frac{3}{2x^{5/2}} - \frac{1}{2x^{3/2}}
\]

12. \( y = u^2; \quad u = \frac{1}{x - 1} \)
\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
= 2u\left(-\frac{1}{(x - 1)^2}\right)
= \left(\frac{2}{x - 1}\right)\left(-\frac{1}{(x - 1)^2}\right)
= -\frac{2}{(x - 1)^3}
\]

14. \( y = u + \frac{1}{u}; \quad u = 5 - 2x \)
\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
= (1 - u^{-2})(-2)
At \( x = 0, u = 5 - 2(0) = 5 \).
\[
\left.\frac{dy}{dx}\right|_{x=0} = \left.\frac{du}{dx}\right|_{u=5} \left.\frac{du}{dx}\right|_{x=0}
= \left(1 - \frac{1}{25}\right)(-2)
= -\frac{48}{25}
\]
16. \( y = u^5 - 3u^2 + 6u - 5; \ u = x^2 - 1 \)

\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (5u^4 - 6u + 6)(2x)
\]

At \( x = 1, \ u = 1^2 - 1 = 0 \).

\[
\frac{dy}{dx}\bigg|_{x=1} = \frac{dy}{du}\bigg|_{u=0} \frac{du}{dx}\bigg|_{x=1} = (6)(2) = 12
\]

18. \( y = 3u^2 - 6u + 2; \ u = \frac{1}{x^2} \)

\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (6u - 6)\left(-\frac{2}{x^3}\right)
\]

At \( x = \frac{1}{3}, \ u = 9 \).

\[
\frac{dy}{dx}\bigg|_{x=\frac{1}{3}} = \frac{dy}{du}\bigg|_{u=9} \frac{du}{dx}\bigg|_{x=\frac{1}{3}} = (48)(-54) = -2,592
\]

20. \( y = \frac{1}{u+1}; \ u = x^3 - 2x + 5 \)

\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = -\frac{1}{(u+1)^2}(3x^2 - 2)
\]

At \( x = 0, \ u = 5 \).

\[
\frac{dy}{dx}\bigg|_{x=0} = \frac{dy}{du}\bigg|_{u=5} \frac{du}{dx}\bigg|_{x=0} = -\frac{1}{36}(-2) = \frac{1}{18}
\]

22. \( f(x) = \frac{1}{\sqrt{5-3x}} = (\sqrt{5-3x})^{-1} \)

\[
f'(x) = -1(\sqrt{5-3x})^{-2}(-3) = \frac{3}{(\sqrt{5-3x})^2}
\]

24. \( f(x) = \sqrt{5x^6 - 12} = (5x^6 - 12)^{1/2} \)

\[
f'(x) = \frac{1}{2}(5x^6 - 12)^{-1/2}(30x^5)
\]

\[
= \frac{15x^5}{\sqrt{5x^6 - 12}}
\]

26. \( f(t) = (3t^4 - 7t^2 + 9)^5 \)

\[
f'(t) = 5(3t^4 - 7t^2 + 9)^4(12t^3 - 14t)
\]

\[
= 10t(6t^2 - 7)(3t^4 - 7t^2 + 9)^4
\]
28. \( f(x) = \frac{2}{(6x^2 + 5x + 1)^2} = 2(6x^2 + 5x + 1)^{-2} \)

\[ f'(x) = -4(6x^2 + 5x + 1)^{-3}(12x + 5) = -\frac{48x + 20}{(6x^2 + 5x + 1)^3} \]

30. \( f(s) = \frac{1}{\sqrt{5s^3 + 2}} = (5s^3 + 2)^{-1/2} \)

\[ f(s) = \left(\frac{-1}{2}\right)(5s^3 + 2)^{-3/2}(15s^2) = -\frac{15s^2}{2(5s^3 + 2)^{3/2}} \]

32. \( f(x) = \frac{2}{3(5x^4 + 1)^2} = \frac{2}{3}(5x^4 + 1)^{-2} \)

\[ f'(x) = -\frac{4}{3}(5x^4 + 1)^{-3}(20x^3) = -\frac{80x^3}{3(5x^4 + 1)^3} \]

34. \( g(x) = \sqrt{1 + \frac{1}{3x}} = \left(1 + \frac{1}{3x}\right)^{1/2} \)

\[ g'(x) = \frac{1}{2}\left(1 + \frac{1}{3x}\right)^{-1/2} \left(-\frac{1}{3x^2}\right) = -\frac{\sqrt{3x}}{6x^2\sqrt{3x + 1}} \]

36. \( f(x) = 2(3x + 1)^4(5x - 3)^2 \)

\[ f'(x) = 2(3x + 1)^4(2)(5x - 3)(5) + 2(4)(3x + 1)^3 (3)(5x - 3)^2 \]

\[ = 4(3x + 1)^3(5x - 3)(45x - 13) \]

38. \( f(y) = \left(\frac{y + 2}{2 - y}\right)^3 \)

\[ f'(y) = 3\left(\frac{y + 2}{2 - y}\right)^2 \frac{(2 - y)(1) - (y + 2)(-1)}{(2 - y)^2} \]

\[ = \frac{12(y + 2)^2}{(2 - y)^4} \]

40. \( F(x) = \frac{(1 - 2x)^2}{(3x + 1)^3} \)

\[ F'(x) = \frac{(3x + 1)^3 2(1 - 2x)(-2) - (1 - 2x)^2 3(3x + 1)^2 (3)}{(3x + 1)^3} \]

\[ = \frac{(1 - 2x)(6x - 13)}{(3x + 1)^4} \]
42. \[ f(x) = \frac{1 - 5x^2}{\sqrt[3]{3 + 2x}} = \frac{1 - 5x^2}{(3 + 2x)^{1/3}} \]
\[ f'(x) = \frac{(3 + 2x)^{1/3}(-10x) - (1 - 5x^2)\frac{1}{3}(3 + 2x)^{-2/3}(2)}{(3 + 2x)^{2/3}} \]
\[ = \frac{-2(25x^2 + 45x + 1)}{3(3 + 2x)^{4/3}} \]

44. \[ f(x) = (9x - 1)^{-1/3} \]
\[ f'(x) = -\frac{1}{3}(9x - 1)^{-4/3}(9) = -3(9x - 1)^{-4/3} \]
At \( x = 1 \), \( y = f(1) = \frac{1}{2} \), \( f'(1) = -\frac{3}{16} \) and the equation of the tangent line is \( y - \frac{1}{2} = -\frac{3}{16}(x - 1) \) or \( y = -\frac{3}{16}x + 1 \frac{11}{16} \).

46. \[ f(x) = (x^2 - 3)^5(2x - 1)^3 \]
\[ f'(x) = (x^2 - 3)^5(2x - 1)^2(2) + (2x - 1)^3(5)(x^2 - 3)^4(2x) \]
At \( x = 2 \), \( y = f(2) = 27 \), \( f'(2) = 594 \) and the equation of the tangent line is \( y - 27 = 594(x - 2) \) or \( y = 594x - 1161 \).

48. \[ f(x) = \left(\frac{x + 1}{x - 1}\right)^3 \]
\[ f'(x) = 3\left(\frac{x + 1}{x - 1}\right)^2 \left(\frac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2}\right) \]
At \( x = 3 \), \( y = f(3) = 8 \), \( f'(3) = -6 \) and the equation of the tangent line is \( y - 8 = -6(x - 3) \) or \( y = -6x + 26 \).

50. \[ f(x) = x^2\sqrt{2x + 3} \]
\[ f'(x) = 2x\sqrt{2x + 3} + x^2 \cdot \frac{1}{2}(2x + 3)^{-1/2}(2) \]
\[ = 2x\sqrt{2x + 3} + \frac{x^2}{\sqrt{2x + 3}} \]
At \( x = -1 \), \( y = f(-1) = (-1)^2\sqrt{2(-1) + 3} = 1 \), \( f'(-1) = 2(-1)\sqrt{2(-1) + 3} + \frac{(-1)^2}{\sqrt{2(-1) + 3}} = -1 \), and the equation of the tangent line is \( y - 1 = -[x - (-1)] \) or \( y = -x \).
52. \( f(x) = x^3(2x^2 + x - 3)^2 \)
\[ f'(x) = x^2(2)(2x^2 + x - 3)(4x + 1) + 3x^2(2x^2 + x - 3)^2 \]
\[ = x^2(x-1)(14x-9)(3+2x)(x+1) \]
The tangent line to the graph of \( f(x) \) is horizontal when \( f'(x) = 0 \) or when \( x = 0, 1, \frac{9}{14}, -\frac{3}{2}, -1 \).

54. \( f(x) = \frac{2x+5}{(1-2x)^3} \)
\[ f'(x) = \frac{(1-2x)^3(2) - (2x+5)(3)(1-2x)^2(-2)}{(1-2x)^6} \]
\[ = \frac{8(4+x)}{(1-2x)^4} \]
The tangent line to the graph of \( f(x) \) is horizontal when \( f'(x) = 0 \) or when \( x = -4 \).

56. \( f(x) = (x-1)^2(2x+3)^3 \)
\[ f'(x) = (x-1)^2(3)(2x+3)^2(2) + (2x+3)^3(2)(x-1) \]
\[ = 10(x-1)(2x+3)^2 \]
The tangent line to the graph of \( f(x) \) is horizontal when \( f'(x) = 0 \) or when \( x = 0, 1, -\frac{3}{2} \).

58. \( f(x) = (7-4x)^2 = (7-4x)(7-4x) \)
By the general power rule
\[ f'(x) = 2(7-4x)(-4) = 32x - 56 \]
By the product rule
\[ f'(x) = (7-4x)(-4) + (7-4x)(-4) \]
\[ = -28 + 16x - 28 + 16x = 32x - 56 \] .

60. \( f(t) = \frac{2}{5t+1} = 2(5t+1)^{-1} \)
\[ f'(t) = -2(5t+1)^{-2}(5) = \frac{-10}{(5t+1)^2} \]
\[ f''(t) = (-2)(-10)(5t+1)^{-3}(5) = \frac{100}{(5t+1)^3} \]

62. \( y = (1-2x^3)^4 \)
\[ y' = 4(1-2x^3)^3(-6x^2) = -24x^2(1-2x^3)^3 \]
\[ y'' = -24x^2(3)(1-2x^3)^2(-6x^2) - 48x(1-2x^3)^3 \]
\[ = 48x(1-2x^3)^2(11x^3 - 1) \]
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64. \( f(u) = \frac{1}{(3u^2 - 1)^2} = (3u^2 - 1)^{-2} \)

\[ f'(u) = -2(3u^2 - 1)^{-3}(6u) = \frac{-12u}{(3u^2 - 1)^3} \]

\[ f''(u) = \frac{6(3u^2 - 1)^2(-12) - 12u(3)(3u^2 - 1)^2(6u)}{(3u^2 - 1)^6} \]

\[ = \frac{12(15u^2 + 1)}{(3u^2 - 1)^4} \]

66. \( C(q) = 0.2q^2 + q + 900 \)

\[ q(t) = t^2 + 100t \]

\[ \frac{dC}{dt} = 2qt = (0.4t + 1)(2t + 100) \]

At \( t = 1 \), \( q = 101 \).

\[ \left. \frac{dC}{dq} \right|_{q = 101} = \left. \frac{dC}{dq} \right|_{q = 101} \frac{dq}{dt} \]

\[ = (41.4)(102) \]

\[ = 4222.8 \]

After 1 hour the manufacturing cost is changing at the rate of $4,222.80 per hour.

68. \( D(p) = \frac{40,000}{p} \)

\[ p(t) = 0.4t^{3/2} + 6.8 \]

Need \( \frac{dD}{dt} \) when \( t = 4 \).

When \( t = 4 \), \( p(4) = 0.4(4)^{3/2} + 6.8 = 10 \).

\[ D(10) = \frac{40,000}{10} = 4000 \]

\[ \frac{dD}{dt} = \frac{dD}{dp} \frac{dp}{dt} \]

Since \( D(p) = \frac{40,000}{p} = 40,000p^{-1}, \quad \frac{dD}{dp} = -40,000p^{-2} = \frac{-40,000}{p^2} \) and \( \frac{dp}{dt} = 0.6t^{1/2} = 0.6\sqrt{t} \).

\[ \frac{dD}{dt} = \frac{-40,000}{p^2} \cdot 0.6\sqrt{t} \]

When \( t = 4 \),

\[ \frac{dD}{dt} = \frac{-40,000}{(4)^{2}} \cdot 0.6\sqrt{4} = -480 \]

\[ 100 \frac{dD}{D(t)} = 100 \frac{-480}{4000} = -12\% \]
\[ 70. \ E = \frac{1}{v} [0.074(v - 35)^2 + 32] \]
\[ E' = \frac{1}{v} (2)(0.074)(v - 35) + \frac{-1}{v^2} [0.074(v - 35)^2 + 32] \]
\[ = \frac{0.074v^2 - 122.65}{v^2} \]

\[ 72. \ (a) \ Q(4) = \frac{4^2 + 2(4) + 3}{2(4) + 1} = \frac{27}{9} = 3 \]
\[ P(3) = 3(3)^2 + 4(3) + 200 = 239 \]
In 4 years the quality-of-life index is expected to be 3, and the population is expected to be 239,000.

\[ (b) \ Q'(t) = \frac{(2t + 1)(2t + 2) - (t^2 + 2t + 3)(2)}{(2t + 1)^2} \]
\[ = \frac{2t^2 + 2t - 4}{(2t + 1)^2} \]
\[ Q'(4) = \frac{2(4)^2 + 2(4) - 4}{(2(4) + 1)^2} = \frac{36}{81} = \frac{4}{9} \]
\[ p'(Q) = 6Q + 4 \]
\[ p'(3) = 6(3) + 4 = 22 \]
\[ \frac{dp}{dt} \bigg|_{t=4} = \frac{dp}{dQ} \bigg|_{Q=3} \frac{dQ}{dt} \bigg|_{t=4} \]
\[ = 22 \left( \frac{4}{9} \right) \]
\[ = 9.778 \]
In 4 years, the population is expected to be increasing at a rate of about 9,778 people per year.

\[ 74. \ (a) \ L(5) = \sqrt{739 + 3(5) - 5^2} = \sqrt{729} = 27 \]
\[ Q(27) = 300(27)^{1/3} = 300(3) = 900 \]
In 5 months, 27 worker-hours will be employed and 900 units will be produced.

\[ (b) \ Q'(L) = \frac{1}{3} (300)L^{\frac{1}{3} - 1} = \frac{100}{L^{2/3}} \]
\[ L'(t) = \frac{1}{2} (739 + 3t - t^2)^{\frac{1}{2} - 1} (3 - 2t) \]
\[ = \frac{3 - 2t}{2\sqrt{739 + 3t - t^2}} \]
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\[ L'(5) = \frac{3 - 2(5)}{2\sqrt{739 + 3(5) - 5^2}} \]
\[ = \frac{-7}{2(27)} \]
\[ = \frac{-7}{54} \]

When \( t = 5 \), \( L = 27 \).

\[ Q'(27) = \frac{100}{27^{2/3}} = \frac{100}{9} \]

\[ \frac{dQ}{dt} \bigg|_{t=5} = \frac{dQ}{dL} \cdot \frac{dL}{dt} \bigg|_{t=5} \]
\[ = \left( \frac{100}{9} \right) \left( \frac{-7}{54} \right) \]
\[ = -1.44 \]

Production will be decreasing at a rate of about 1.44 units per month.

76. \( V(N) = \left( \frac{3N + 430}{N + 1} \right)^{2/3} \]

\( N(t) = \sqrt{t^2 - 10t + 45} = (t^2 - 10t + 45)^{1/2} \)

(a) \( N(9) = \sqrt{(9)^2 - 10(9) + 45} \)

\( = 6 \) hours per day

\( V(6) = \left( \frac{3(6) + 430}{6 + 1} \right)^{2/3} \)

\( = 16 \) or \$16,000.

(b) \( \frac{dV}{dt} = \frac{dV}{dN} \cdot \frac{dN}{dt} \)

\[ \frac{dV}{dN} = \frac{2}{3} \left( \frac{3N + 430}{N + 1} \right)^{-1/3} \cdot \frac{(N + 1)(3) - (3N + 430)(1)}{(N + 1)^2} \]
\[ = \frac{2(N + 1)^{1/3}}{3(3N + 430)^{1/3}} \cdot \frac{-427}{(N + 1)^2} \]
\[ = -\frac{854}{3(3N + 430)^{1/3}(N + 1)^{5/3}} \]

\[ \frac{dN}{dt} = \frac{1}{2} \left( t^2 - 10t + 45 \right)^{-1/2} (2t - 10) \]
\[ = \frac{t - 5}{(t^2 - 10t + 45)^{1/2}} \]

Using \( t = 9 \) and \( N = 6 \).

\[ \frac{dV}{dt} = \frac{854}{3(3(6) + 430)^{1/3}(6 + 1)^{5/3}} \cdot \frac{9 - 5}{[(9)^2 - 10(9) + 45]^{1/2}} \]
\[ = -0.968 \text{ thousand} \]
or −968 dollars per month

Since $\frac{dV}{dt}$ is negative when $t = 9$, the value will be decreasing.

78. (a) $\frac{dA}{dr} = (520)(10,000)\left(1 + \frac{r}{52}\right)^{519} \cdot \frac{1}{52}$

$= 100,000\left(1 + \frac{r}{52}\right)^{519}$

(b) % rate of change

$= 100 \frac{dA/dr}{A}$

$= 100 \frac{100,000\left(1 + \frac{r}{52}\right)^{519}}{10,000\left(1 + \frac{r}{52}\right)^{520}}$

$= \frac{1,000}{1 + \frac{r}{52}}$

When $r = 0.05$ the percent rate of change with respect to $A$ is $\frac{1,000}{1 + \frac{0.05}{52}} = 999.039$.

80. (a) $v(t) = (2t + 9)^2(8-t)^3$ \quad 0 \leq t \leq 5

$a(t) = (2t + 9)^2(3)(8-t)^2(-1) + (8-t)^3(2)(2t+9)(2)$

$= 5(2t + 9)(8-t)^2(1 - 2t)$

(b) The object is stationary when $v(t) = 0$. The function $(2t + 9)^2(8-t)^3$ is 0 when $t = \frac{-9}{2}$ and $t = 8$. Since neither value falls in the interval $0 \leq t \leq 5$, the object is never stationary.

(c) $a(t) = 0$ when $t = \frac{1}{2}$. At this time $v\left(\frac{1}{2}\right) = 42,187.5$.

(e) The object is speeding up for $0 \leq t \leq \frac{1}{2}$. 
82. (a) \( f(x) = L(u(x)); u(x) = x^2 \)
\[ f'(x) = L'(u(x))u'(x) \]
\[ = \frac{1}{u(x)} (2x) \]
\[ = \frac{1}{x^2} (2x) \]
\[ = 2 \]
\[ \frac{1}{x} \]

(b) \( f(x) = L(u(x)); u(x) = \frac{1}{x} \)
\[ f'(x) = L'(u(x))u'(x) \]
\[ = \frac{1}{u(x)} \left( -\frac{1}{x^2} \right) \]
\[ = \frac{1}{x} \left( -\frac{1}{x^2} \right) \]
\[ = \frac{1}{x} \]

(c) \( f(x) = L(u(x)); u(x) = \frac{2}{3\sqrt{x}} \)
\[ f'(x) = L'(u(x))u'(x) \]
\[ = \frac{1}{u(x)} \left( -\frac{1}{3x^{3/2}} \right) \]
\[ = \frac{3\sqrt{x}}{2} \left( -\frac{1}{3x^{3/2}} \right) \]
\[ = \frac{-1}{2x} \]

(d) \( f(x) = L(u(x)); u(x) = \frac{2x+1}{1-x} \)
\[ f'(x) = L'(u(x))u'(x) \]
\[ = \frac{1}{u(x)} \left( \frac{3}{(1-x)^2} \right) \]
\[ = \frac{1-x}{2x+1} \left( \frac{3}{(1-x)^2} \right) \]
\[ = \frac{3}{(2x+1)(1-x)} \]

84. \( \frac{d}{dx} [h(x)]^3 \)
\[ = \frac{d}{dx} h(x)[h(x)]^2 \]
\[ = h(x)2(h(x))h'(x) + [h(x)]^2 h'(x) \]
\[ = 2[h(x)]^2 h'(x) + [h(x)]^2 h'(x) \]
\[ = 3[h(x)]^2 h'(x) \]

86. \( f'(0) \) does not exist while \( f'(4.3) \approx 16.63 \). The graph of \( f(x) \) has one horizontal tangent when \( x \approx 0.50938 \).

2.5 Marginal Analysis and Approximations Using Increments

2. (a) \( C(x) = \frac{1}{4} x^2 + 3x + 67 \)
\[ C'(x) = \frac{1}{2} x + 3 \]
\[ R'(x) = 9 - \frac{2}{5} x \]

(b) \( C'(3) = \frac{1}{2} (3) + 3 = 4.50 \)

(c) \( C(4) - C(3) \)
\[ = (4 + 12 + 67) - \left( \frac{9}{4} + 9 + 67 \right) = 4.75 \]

(d) \( R'(3) = 9 - \frac{2}{5} (3) = 7.80 \)

(e) \( R(4) - R(3) \)
\[ = \left( \frac{36 - 16}{5} \right) - \left( \frac{27 - 9}{5} \right) = 7.60 \]
4. (a) \( C(x) = \frac{5}{9}x^2 + 5x + 73 \)
\( R(x) = xp(x) = x\left[ -x^2 - 2x + 33 \right] = -x^3 - 2x^2 + 33x \)
\( C'(x) = \frac{10}{9}x + 5 \)
\( R'(x) = -3x^2 - 4x + 33 \)

(b) \( C'(3) = \frac{10}{9}(3) + 5 = 8.33 \)

(c) \( C(4) - C(3) = \left( \frac{80}{9} + 20 + 73 \right) - (5 + 15 + 73) = 8.89 \)

(d) \( R'(3) = -27 - 12 + 33 = -6.00 \)

(e) \( R(4) - R(3) = (-64 - 32 + 132) - (-27 - 18 + 99) = -18.00 \)

6. (a) \( C(x) = \frac{2}{7}x^2 + 65 \)
\( R(x) = xp(x) = x\left[ \frac{12 + 2x}{3 + x} \right] = \frac{12x + 2x^2}{3 + x} \)
\( C'(x) = \frac{4}{7}x \)
\( R'(x) = \frac{2(x^2 + 6x + 18)}{(x + 3)^2} \)

(b) \( C'(3) = \frac{4}{7}(3) = 1.71 \)

(c) \( C(4) - C(3) = \left( \frac{32}{7} + 65 \right) - \left( \frac{18}{7} + 65 \right) = 2.00 \)

(d) \( R'(3) = 2\left( \frac{45}{36} \right) = 2.50 \)

(e) \( R(4) - R(3) = \left( \frac{80}{7} \right) - 9 = 2.43 \)

8. \( f(x) = \frac{x}{x+1} - 3 \)
\( f'(x) = \frac{1}{(x+1)^2} \)
\( f'(4) = \frac{1}{25} \) and \( \Delta x = 3.8 - 4 = -0.2 \) so
\( \Delta f = \frac{1}{25}(-0.2) = -0.008 \)
Thus \( f(x) \) will decrease by about 0.008.

10. \( f(x) = 3x + \frac{2}{x} \)
\( f'(x) = 3 - \frac{2}{x^2} \)
\( f'(5) = 3 - \frac{2}{25} = 2.92 \) and
\( \Delta x = 5 - 4.6 = -0.4 \) so
\( \Delta f = 2.92(-0.4) = -1.168 \)
Thus \( f(x) \) will decrease by about 1.168.

12. (a) \( R(q) = 240q - 0.05q^2 \)
\( R'(q) = 240 - 0.1q \)
At a production level of 80 units per month the marginal revenue is \( R'(80) = 232 \) so the additional revenue if production is increased by one unit is approximately $232.

(b) \( R(81) - R(80) = $19,111.95 - $18,880.00 = $231.95 \)

14. \( Q(t) = 0.05t^2 + 0.1t + 3.4 \)
\( Q'(t) = 0.1t + 0.1 \)
\( Q'(0) = 0.1, \Delta t = 0.5 \) (6 months)
The approximate change in carbon monoxide level will be \( \Delta Q = Q'(0)\Delta t = 0.05 \) ppm.

16. \( C(q) = 0.1q^3 - 0.5q^2 + 500q + 200 \)
\( C'(q) = 0.3q^2 - q + 500 \)
\( C'(4) = 500.8, \Delta q = 4.1 - 4 = 0.1 \)
The approximate change in cost will be \( \Delta C = C'(4)\Delta q = $50.08 \).
18. \( f(x) = -x^3 + 6x^2 + 15x \)
    \[ f'(x) = -3x^2 + 12x + 15 \]
    At 9:00 A.M., \( x = 1 \)
    \[ f'(1) = 24 \]
    Further, \( \Delta x = 0.25 \) (one quarter hour) so the approximate change in radio production from 9:00 to 9:15 A.M. will be
    \[ \Delta f = f'(1)\Delta x = 6 \text{ radios.} \]

20. \( Q(L) = 60,000L^{1/3} \)
    \[ Q'(L) = \frac{20,000}{L^{2/3}} \]
    \( Q'(1,000) = 200, \Delta L = 940 - 1,000 = -60 \)
    The approximate effect on output will be
    \[ \Delta Q = Q'(1,000)\Delta L = -12,000, \]
    that is, a decrease of about 12,000 units.

22. (a) \( P(t) = -t^3 + 9t^2 + 48t + 200 \)
    \[ P'(t) = -3t^2 + 18t + 48 \]
    (b) \( R'(t) = -6t + 18 \)
    (c) \( R'(3) = 0 \) and \( \Delta x = \frac{1}{12} \) (one month)
    so \( \Delta R = 0 \). There is no expected change in the population growth rate during the first month of the fourth year.

24. \( Q(L) = 300L^{2/3} \)
    \[ Q'(L) = \frac{200}{L^{1/3}}, \quad Q'(512) = 25 \]
    We seek \( \Delta L \) so that
    \[ 12.5 = \Delta Q = Q'(512)\Delta L = 25\Delta L, \]
    so \( \Delta L = 0.5 \) more worker-hours are needed.

26. A 1% increase in \( r \) means \( \Delta r = 0.01r \) or \( \frac{\Delta r}{r} = 0.01 \). For the surface area,
    \[ S = 4\pi r^2, \]
    \[ \Delta S = \frac{dS}{dr}\Delta r = 8\pi r\Delta r \]
    \[ = 8\pi r(0.01)r \]
    \[ = 0.08\pi r^2 \]
    \[ = 0.02(4\pi r^2) \]
    \[ = 0.02S \]
    So the surface area increases by approximately 2%.

28. The inner volume of the balloon is given by \( V = \frac{4}{3}\pi(0.01)^3 = 4.189 \times 10^{-6} \) cubic millimeters. The volume of the balloon skin can be approximated as
    \[ \Delta V = V'(r)\Delta r = 4\pi(0.01)^2(0.0005) \]
    \[ = 6.283 \times 10^{-7} \text{ cubic millimeters.} \]
    The total volume inserted is
    \[ V + \Delta V = 4.817 \times 10^{-6} \text{ mm}^3. \]

30. (a) \[ \frac{\Delta V}{V} = \frac{V'\Delta R}{V} = \frac{4kR^3\Delta R}{kR^4} = \frac{4\Delta R}{R} \]
    If \( \frac{\Delta R}{R} = 0.05 \) then
    \[ \Delta V = 4(0.05) = 0.20 \text{ or the volume increases by about 20%.} \]
    (b) Writing exercise—answers will vary.

32. \[ 100\frac{\Delta R}{R} = 100\frac{R'\Delta T}{R} = 100\frac{4kT^3\Delta T}{kT^4} = 400\frac{\Delta T}{T} \]
    If \( \frac{\Delta T}{T} = 0.02 \) then the percentage change in \( R \) is approximately 8%. 
34. (a) The root is approximately 1.465571.

(b) For \( f(x) = x^3 - x^2 - 1 \), the iterative formula for Newton’s method is
\[
x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} = x_{n-1} - \frac{x_{n-1}^3 - x_{n-1}^2 - 1}{3x_{n-1}^2 - 2x_{n-1}}
\]
Beginning with \( x_0 = 1 \), the sequence of approximations to the root are
\[
x_0 = 1 \quad x_1 = 2 \quad x_2 = 1.625 \quad x_3 = 1.485786 \quad x_4 = 1.465956 \quad x_5 = 1.465571 \quad x_6 = 1.465571
\]
The sixth estimate agrees with the fifth to at least four decimal places.

(c) Answers will vary based on the accuracy of the estimate in part (a).

36. (a) Suppose \( N \) is a fixed number and let \( f(x) = x^2 - N \). Then \( f'(x) = 2x \) and Newton’s method becomes
\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n}
\]
\[
= x_n - \frac{x_n}{2} + \frac{N}{2x_n} = \frac{1}{2} \left( x_n + \frac{N}{x_n} \right)
\]

(b) Writing exercise—answers will vary.

2.6 Implicit Differentiation and Related Rates

2. (a) Differentiating both sides of
\[
5x - 7y = 3
\]
with respect to \( x \) yields
\[
5 - 7 \frac{dy}{dx} = 0.
\]
Solving for \( \frac{dy}{dx} \) gives
\[
\frac{dy}{dx} = \frac{5}{7}.
\]

(b) Solving for \( y \) gives \( y = \frac{5}{7}x - \frac{3}{7} \) so
\[
\frac{dy}{dx} = \frac{5}{7}.
\]

4. (a) Differentiating both sides of
\[
x^2 + y^3 = 12
\]
with respect to \( x \) yields
\[
2x + 3y^2 \frac{dy}{dx} = 0.
\]
Solving for \( \frac{dy}{dx} \) gives \( \frac{dy}{dx} = -\frac{2x}{3y^2} \).

(b) Solving for \( y \) gives \( y = (12 - x^2)^{1/3} \) so
\[
\frac{dy}{dx} = \frac{-2x}{3(12 - x^2)^{2/3}} = -\frac{2x}{3y^2}.
\]

6. (a) Differentiating both sides of
\[
x + \frac{1}{y} = 5
\]
with respect to \( x \) yields
\[
1 - \frac{1}{y^2} \frac{dy}{dx} = 0.
\]
Solving for \( \frac{dy}{dx} \) gives \( \frac{dy}{dx} = y^2 \).

(b) Solving for \( y \) gives \( y = \frac{1}{5-x} \) so
\[
\frac{dy}{dx} = \frac{1}{(5-x)^2} = y^2.
\]

8. (a) Differentiating both sides of
\[
xy + 2y = x^2
\]
with respect to \( x \) yields
\[
y + x \frac{dy}{dx} + 2 \frac{dy}{dx} = 2x.
\]
Solving for \( \frac{dy}{dx} \) gives
\[
\frac{dy}{dx} = \frac{2x - y}{x + 2}.
\]
(b) Solving for \( y \) gives \( y = \frac{x^2}{x + 2} \) so
\[
\frac{dy}{dx} = \frac{2x(x + 2) - x^2}{(x + 2)^2} (1)
\]
\[
= \frac{2x - \frac{x^2}{x+2}}{x + 2}
\]
\[
= \frac{2x - y}{x + 2}.
\]

10. \( x^2 + y = x^3 + y^2 \)
\[
2x + \frac{dy}{dx} = 3x^2 + 2y \frac{dy}{dx}
\]
\[
(1 - 2y) \frac{dy}{dx} = 3x^2 - 2x
\]
\[
\frac{dy}{dx} = \frac{3x^2 - 2x}{1 - 2y}
\]

12. \( 5x - x^2y^3 = 2y \)
\[
5 - x^2(3y^2) \frac{dy}{dx} - 2xy^3 = 2 \frac{dy}{dx}
\]
\[
(2 + 3x^2y^2) \frac{dy}{dx} = 5 - 2xy^3
\]
\[
\frac{dy}{dx} = \frac{5 - 2xy^3}{2 + 3x^2y^2}
\]

14. \( \frac{1}{x} + \frac{1}{y} = 1 \)
\[
- \frac{1}{x^2} - \frac{1}{y^2} \frac{dy}{dx} = 0
\]
\[
\frac{dy}{dx} = -\frac{y^2}{x^2}
\]

16. \( \sqrt{2x} + y^2 = 4 \)
\[
\frac{\sqrt{2}}{2\sqrt{x}} + 2y \frac{dy}{dx} = 0
\]
\[
\frac{dy}{dx} = -\frac{1}{2y\sqrt{2x}}
\]
18. \[ y^2 + 3xy - 4x^2 = 9 \]
\[
2y \frac{dy}{dx} + 3x \frac{dy}{dx} + 3y - 8x = 0
\]
\[
(2y + 3x) \frac{dy}{dx} = 8x - 3y
\]
\[
\frac{dy}{dx} = \frac{8x - 3y}{2y + 3x}
\]

20. \[ (x - 2y)^2 = y \]
\[
2(x - 2y) \left(1 - 2 \frac{dy}{dx}\right) = \frac{dy}{dx}
\]
\[
2(x - 2y) = \frac{dy}{dx} + 4(x - 2y) \frac{dy}{dx}
\]
\[
\frac{dy}{dx} = \frac{2(x - 2y)}{1 + 4(x - 2y)}
\]
\[
= \frac{2x - 4y}{1 + 4x - 8y}
\]

22. \[ (3xy^2 + 1)^4 = 2x - 3y \]
\[
4(3xy^2 + 1)^3 \left(6xy \frac{dy}{dx} + 3y^2\right) = 2 - 3 \frac{dy}{dx}
\]
\[
24xy(3xy^2 + 1)^3 \frac{dy}{dx} + 3 \frac{dy}{dx} = 2 - 12y^2(3xy^2 + 1)^3
\]
\[
\frac{dy}{dx} = \frac{2 - 12y^2(3xy^2 + 1)^3}{24xy(3xy^2 + 1)^3 + 3}
\]

24. \[ x^2 - y^3 = 2x \]
\[
2x - 3y^2 \frac{dy}{dx} = 2
\]
\[
\frac{dy}{dx} = \frac{2 - 2x}{-3y^2}
\]
At (1, -1) the slope is \[
\frac{2 - 2(1)}{-3(-1)^2} = 0
\]
and the equation of the tangent line is \(y = -1\).

26. \[ \frac{1}{x} - \frac{1}{y} = 2 \]
\[
-\frac{1}{x^2} + \frac{1}{y^2} \frac{dy}{dx} = 0
\]
\[
\frac{dy}{dx} = \frac{y^2}{x^2}
\]
At \(\left(\frac{1}{4}, \frac{1}{2}\right)\) the slope is \[
\left(\frac{1}{4}\right)^2 = 4
\]
and the equation of the tangent line is \(y = 4x - \frac{1}{2}\).
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28. \[ \frac{d}{dx} \left( x^2 y^3 - 2xy = 6x + y + 1 \right) \]
\[ 3x^2 y^2 \frac{dy}{dx} + 2xy^3 - 2x \frac{dy}{dx} - 2y = 6 + \frac{dy}{dx} \]
\[ \frac{dy}{dx} = \frac{2y - 2xy^3 + 6}{3x^2 y^2 - 2x - 1} \]
At \( (0, -1) \) the slope is \( \frac{-2 - 0 + 6}{0 - 0 - 1} = -4 \) and the equation of the tangent line is \( y = -4x - 1 \).

30. \[ (x^2 + 2y)^3 = 2xy^2 + 64 \]
\[ 3(x^2 + 2y)^2 \left( 2x + 2 \frac{dy}{dx} \right) = 4xy \frac{dy}{dx} + 2y^2 \]
At \( (0, 2) \) this equation becomes \( 96 \frac{dy}{dx} = 8 \) so the slope at \( (0, 2) \) is \( \frac{1}{12} \) and the equation of the tangent line is \( y = \frac{1}{12} x + 2 \).

32. (a) \[ x^2 + xy + y^3 = 3 \]
\[ 2x + x \frac{dy}{dx} + y \cdot 1 + \frac{dy}{dx} = 0 \]
\[ \frac{dy}{dx} = \frac{-2x - y}{x + 1} \]
\[ \frac{-2x - y}{x + 1} = 0 \text{ when } -2x - y = 0, \text{ or } y = -2x. \]
Substituting in the original equation,
\[ x^2 - 2x^2 - 2x = 3 \]
\[ 0 = x^2 + 2x + 3 \]
Since there are no real solutions, there are no horizontal tangents.

(b) \[ x + 1 = 0 \text{ when } x = -1. \]
When \( x = -1, 1 - y + y = 3 \)
So no such \( y \) exists and there are no vertical tangents.

34. \[ \frac{y - x}{x} \]
\[ \frac{x \frac{dy}{dx} - y}{x^2} - \frac{y - x \frac{dy}{dx}}{y^2} = 0 \text{ which simplifies to } \frac{dy}{dx} = \frac{y}{x}. \]
From the equation of the curve, there can be no points on the curve having either \( x = 0 \) or \( y = 0 \).
Thus there are no points where the numerator or denominator of the derivative is 0. There are no horizontal or vertical tangents to this curve.
36. (a) \[ x^2 - xy + y^2 = 3 \]
\[ 2x - x \frac{dy}{dx} - y + 2y \frac{dy}{dx} = 0 \]
\[ \frac{dy}{dx} = \frac{y - 2x}{2y - x} \]
For the tangent line to be horizontal we must have \( \frac{dy}{dx} = 0 \) and \( y = 2x \) at such a point.
Substituting this expression into the equation of the curve gives \( x^2 - x(2x) + (2x)^2 = 3x^2 = 3 \) so \( x = \pm 1 \). Since \( y = 2x \), the points where the tangent is horizontal are (1, 2) and (-1, -2).

(b) The tangent line will be vertical when the denominator in the derivative is 0 while the numerator is not 0. The denominator is 0 at points where \( x = 2y \). Substitution into the original equation gives \( 3y^2 = 3 \) and the points where the tangent line is vertical are (2, 1) and (-2, -1).

38. \[ xy + y^2 = 1 \]
\[ x \frac{dy}{dx} + y + 2y \frac{dy}{dx} = 0 \]
\[ \frac{dy}{dx} = \frac{-y}{x + 2y} \]
\[ \frac{d^2 y}{dx^2} = \frac{\left(x + 2y\right) \left(-\frac{dy}{dx}\right) - (-y) \left(1 + 2 \frac{dy}{dx}\right)}{(x + 2y)^2} \]
\[ = \frac{\left(x + 2y\right) \left(\frac{y}{x + 2y}\right) - (-y) \left(1 + 2 \left(\frac{-y}{x + 2y}\right)\right)}{(x + 2y)^2} \]
\[ = \frac{2y(x + y)}{(x + 2y)^3} \]

40. \[ Q = 0.06x^2 + 0.14xy + 0.05y^2 \]
The goal is keep \( Q \) constant hence upon differentiating
\[ 0 = \frac{dQ}{dx} = 0.12x + 0.14x \frac{dy}{dx} + 0.14y + 0.10y \frac{dy}{dx} \]
\[ \frac{dy}{dx} = \frac{-0.12x - 0.14y}{0.14x + 0.10y} = \frac{-6x - 7y}{7x + 5y} \]
Use the approximation formula \( \Delta y \approx \frac{dy}{dx} \Delta x \) with \( x = 60 \), \( y = 300 \) and \( \Delta x = 1 \).
\[ \Delta y \approx \frac{-6(60) - 7(300)}{7(60) + 5(300)}(1) = -1.28125 \]
To maintain output at the current level decrease the unskilled labor by 1.28125 hours.
42. $x^2 + 3px + p^2 = 79$

$$2x\frac{dx}{dt} + 3p\frac{dp}{dt} + 3x\frac{dp}{dt} + 2p\frac{dp}{dt} = 0$$ or $$\frac{dx}{dt} = -\frac{3x - 2p}{2x + 3p} \frac{dp}{dt}$$

When $p = 5$, the demand $x$ satisfies $p = 5$

$$x^2 + 3(5)x + 5^2 = 79$$ or $$x^2 + 15x - 54 = (x+18)(x-3) = 0$$ so $x = 3$. Give $\frac{dp}{dt} = 0.30$.

$$\frac{dx}{dt} = -\frac{19}{21}(0.30) = -0.27143$$ or demand is decreasing at the rate 27.143 units per month.

44. (a) Since $V = s^3$, $s^3 = 125,000$ and $s = 50$. Differentiating with respect to $t$ $V' = 3s^2 s'$

At the present time $V' = -1,000$ and $s = 50$ so $s' = \frac{V'}{3s^2} = -\frac{1,000}{3(50)^2} = -\frac{2}{15}$ cm per hour.

(b) $S = 6s^2$

$$S' = 12ss' = 12(50)\left(-\frac{2}{15}\right) = -80 \text{ cm}^2 \text{ per hour}$$

46. $Q(p) = p^2 + 4p + 900$

$Q' = (2p + 4)p'$ where the derivatives are with respect to $t$. At the time in question $p = 50$ and $p' = 1.5$ so the pollution level is changing at the rate of $Q' = (2(50) + 4)(1.5) = 156$ units per year.

48. $PV = C$

$PV' + VP' = 0$ or $P' = -\frac{PV'}{V}$ where prime denotes differentiation with respect to $t$. At the time in question $P' = -\frac{70(12)}{40} = -21$. The pressure is decreasing at the rate of 21 lb/in.$^2$/sec.

50. $s = 1.1w^{0.2}$

$$\frac{ds}{dt} = 0.22w^{-0.8} \frac{dw}{dt}$$

When $w = 11$ and $\frac{dw}{dt} = 0.02$, $\frac{ds}{dt} = 0.22(11)^{-0.8}(0.02) = 0.000646$ meters per second per day

52. $A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

When $r = 200$ and $\frac{dr}{dt} = 20$

$$\frac{dA}{dt} = 2\pi(200)20 = 8,000\pi \text{ or roughly 25,133 square feet per hour.}$$
54. \( PV^{1.4} = C \)

Differentiating with respect to \( t \) yields

\[ P(1.4V^{-0.4})V' + P'V^{1.4} = 0 \]

\[ V' = \frac{-P'V^{1.8}}{1.4P} \]

Given \( V = 5 \), \( P = 0.6 \) and \( P' = 0.23 \)

\[ V' = \frac{-0.23(5^{1.8})}{1.4(0.6)} = -4.961. \]

\( V \) is decreasing at roughly 4.96 m\(^3\) per sec.

56. \( V = \frac{\pi}{3} H(R^2 + rR + r^2) \)

\[ \frac{dV}{dt} = \frac{\pi}{3} \left[ H \left( 2R \frac{dR}{dt} + r \frac{dR}{dt} + R \frac{dr}{dt} + 2r \frac{dr}{dt} \right) + \frac{dH}{dt}(R^2 + rR + r^2) \right] \]

Substituting \( r = 2 \), \( R = 3 \), \( H = 15 \) and \( \frac{dr}{dt} = \frac{4}{12} \), \( \frac{dR}{dt} = \frac{5}{12} \), \( \frac{dH}{dt} = \frac{9}{12} \) yields

\[ \frac{dV}{dt} = \frac{397\pi}{12} \approx 103.93 \text{ cubic feet per year.} \]

58. \( Q = 3u^2 + \frac{2u + 3v}{(u + v)^2} \)

The goal is to keep \( Q \) constant hence upon differentiating

\[ 0 = \frac{dQ}{du} = 6u + \frac{\left[ (u + v)^2 \left( 2 + 3\frac{dv}{du} \right) - 2(2u + 3v)(u + v) \left( 1 + \frac{dv}{du} \right) \right]}{(u + v)^4} \]

When \( u = 10 \) and \( v = 25 \), solving the above for \( \frac{dv}{du} \) gives \( \frac{dv}{du} = \frac{514476}{17} \).

Use the approximation formula \( \Delta v \approx \frac{dv}{du} \Delta u \) with \( \Delta u = -0.7 \).

\[ \Delta v \approx \frac{514476}{17}(-0.7) = 21184.3. \]

To maintain output at the current level decrease the unskilled labor by 21,184 units.

60. (a) \( x^2 + y^2 = 6y - 10 \)

\[ x^2 + y^2 - 6y + 9 = -10 + 9 \]

\[ x^2 + (y - 3)^2 = -1 \]

Since the sum of two squares cannot be negative, there are no points \((x, y)\) that satisfy this equation.
(b) \[ 2x + 2y \frac{dy}{dx} = 6 \frac{dy}{dx} \]
\[ 2(3 - y) \frac{dy}{dx} = 2x \]
\[ \frac{dy}{dx} = \frac{x}{3 - y} \]

62.

For the tangent line to be horizontal at a point, we must have \( y = 5x \) so that \( \frac{dy}{dx} = 0 \). Substituting \( y = 5x \) in \( 5x^2 - 2xy + 5y^2 = 8 \) gives \( 5x^2 - 2x(5x) + 5(5x)^2 = 120x^2 = 8 \) yielding \( x = \pm \frac{1}{\sqrt{15}} \) and \( y = \pm \frac{5}{\sqrt{15}} \). There are two horizontal tangents with equations \( y = \frac{5}{\sqrt{15}} \) and \( y = -\frac{1}{\sqrt{15}} \).

64. (a) \( x^3 + y^3 = 3xy \)
\[ 3x^2 + 3y^2 \frac{dy}{dx} = 3x \frac{dy}{dx} + 3y \]
\[ \frac{dy}{dx} = \frac{y - x^2}{y^2 - x} \]

A point where the tangent is horizontal must satisfy \( \frac{dy}{dx} = 0 \) or \( y = x^2 \). Substituting into the equation of the curve gives
\[ x^3 + (x^2)^3 = 3x(x^2) \]
\[ x^6 - 2x^3 = x^3(x^3 - 2) = 0 \]

yielding \( x = 0 \) and \( x = \sqrt[3]{2} \). If \( x = 0 \), then \( y = 0 \) and the derivative is undefined. When \( x = \sqrt[3]{2} \), then \( y = \sqrt[3]{4} \) and so the equation of the horizontal tangent is \( y = \sqrt[3]{4} \).
(b) Substituting \( y = x \) into the equation of the curve gives
\[
\begin{align*}
x^3 + x^3 &= 3x^2 \\
2x^3 - 3x^2 &= x^2(2x - 3)
\end{align*}
\]
so \( x = \frac{3}{2} \) and \( y = \frac{3}{2} \). The slope at this point is
\[
\left( \frac{3}{2} \right)^{\frac{3}{2}} = -1
\]
and the equation of the tangent line is \( y = -x + 3 \).

(c)

Review Exercises

2. \( f(x) = \frac{1}{x - 2} \)
As \( h \to 0 \) this difference quotient approaches \( \frac{-1}{(x - 2)^2} \), so \( f'(x) = \frac{-1}{(x - 2)^2} \)

4. \( f(x) = x^3 - \frac{1}{3}x^5 + 2\sqrt{x} - \frac{3}{x} + \frac{1 - 2x}{x^3} \)
\[
= x^3 - \frac{1}{3}x^{-5} + 2x^{1/2} - 3x^{-1} + x^{-3} - 2x^{-2}
\]
\( f'(x) = 3x^2 + \frac{5}{3}x^{-6} + x^{-1/2} + 3x^{-2} - 3x^{-4} + 4x^{-3} \)
\[
= 3x^2 + \frac{5}{3x^6} + \frac{1}{\sqrt{x}} + \frac{3}{x^2} - \frac{3}{x^4} + \frac{4}{x^3}
\]

6. \( y = (x^3 + 2x - 7)(3 + x - x^2) \)
\[
\frac{dy}{dx} = (x^3 + 2x - 7)(1 - 2x) + (3 + x - x^2)(3x^2 + 2)
\]
\[
= -5x^4 + 4x^3 + 3x^2 + 18x - 1
\]

8. \( f(x) = \sqrt{x^2 + 1} = \left( x^2 + 1 \right)^{1/2} \)
\( f'(x) = \frac{1}{2} \left( x^2 + 1 \right)^{-1/2} (2x) = \frac{x}{\sqrt{x^2 + 1}} \)
10. \( y = \left( \frac{x+1}{1-x} \right)^2 \)

\[
\frac{dy}{dx} = 2 \left( \frac{x+1}{1-x} \right) \frac{d}{dx} \left( \frac{x+1}{1-x} \right)
\]

\[
= 2 \frac{x+1}{1-x} \cdot \frac{(1-x) - (x+1)(-1)}{(1-x)^2}
\]

\[
= 2 \frac{x+1}{1-x} \cdot \frac{2}{(1-x)^2}
\]

\[
= \frac{4(x+1)}{(1-x)^3}
\]

12. \( f(x) = \frac{(3x+1)^3}{(1-3x)^4} \)

\[
f'(x) = \frac{1}{(1-3x)^8}
\]

\[
\left[ (1-3x)^4 \frac{d}{dx} (3x+1)^3 - (3x+1)^3 \frac{d}{dx} (1-3x)^4 \right]
\]

\[
= \frac{1}{(1-3x)^8} \left\{ (1-3x)^4 \left[ 3(3x+1)^2 (3) \right] - (3x+1)^3 \left[ 4(1-3x)^3 (-3) \right] \right\}
\]

\[
= \frac{3(3x+1)^2 (3x+7)}{(1-3x)^5}
\]

14. \( f(x) = x^2 - 3x + 2 \)

\[
f'(x) = 2x - 3
\]

\( f(1) = 0 \). The slope of the tangent line at \((1,0)\) is \( m = f'(1) = -1 \). The equation of the tangent line is \( y - 0 = -(x - 1) \) or \( y = -x + 1 \)

16. \( f(x) = \frac{x}{x^2 + 1} \)

\[
f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{1-x^2}{(x^2 + 1)^2}
\]

\( f(0) = 0 \). The slope of the tangent line at \((0,0)\) is \( m = f'(0) = 1 \). The equation of the tangent line is \( y - 0 = x - 0 \) or \( y = x \)

18. (a) The rate of change of

\( f(t) = t^3 - 4t^2 + 5t\sqrt{t} - 5 \)

\[
= t^3 - 4t^2 + 5t^{3/2} - 5
\]

is \( f'(t) = 3t^2 - 8t + \frac{15}{2} t^{1/2} \)

at any value of \( t \geq 0 \) and when \( t = 4 \), \( f'(4) = 48 - 32 + \frac{15}{2} (2) = 31 \).
(b) The rate of change of \( f(t) = \frac{2t^2 - 5}{1 - 3t} \)

is \( f'(t) = \frac{(1 - 3t)4t - (2t^2 - 5)(-3)}{(1 - 3t)^2} \)

\[ = \frac{-6t^2 + 4t - 15}{(1 - 3t)^2} \]

at any value of \( t \neq \frac{1}{3} \). When \( t = -1 \),

\[ f'(-1) = \frac{-6 - 4 - 15}{4^2} = -\frac{25}{16}. \]

20. (a) \( f(t) = t^2 - 3t + \sqrt{t}; f(4) = 16 - 12 + 2 = 6 \)

\[ f'(t) = 2t - 3 + \frac{1}{2\sqrt{t}}; \]

\( f'(4) = 8 - 3 + \frac{1}{4} = 21 \frac{4}{4} \)

\[ 100 \frac{f'(t)}{f(t)} = 100 \frac{21}{6} = 87.5 \]

The percentage rate of change is 87.5%.

(b) \( f(t) = \frac{t}{t-3}; f(4) = \frac{4}{4-3} = 4 \)

\[ f'(t) = \frac{(t-3) - t}{(t-3)^2} = -\frac{3}{(t-3)^2}; \]

\( f'(4) = -\frac{3}{(4-3)^2} = -3 \)

\[ 100 \frac{f'(t)}{f(t)} = 100 \frac{-3}{4} = -75 \]

The percentage rate of change is -75%.

22. (a) \( y = 5u^2 + u - 1, \; u = 3x + 1 \)

\[ \frac{dy}{du} = 10u + 1, \; \frac{du}{dx} = 3, \]

\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (10u + 1)(3) = 3(30x + 11) \]

(b) \( y = \frac{1}{u^2}, \; u = 2x + 3, \; \frac{dy}{du} = -\frac{2}{u^3}, \; \frac{du}{dx} = 2 \)

\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\frac{4}{(2x+3)^3} \]

24. (a) \( y = u - u^2; \; u = x - 3 \)

When \( x = 0, \; u = -3 \).

\[ \frac{dy}{dx} \bigg|_{x=0} = \frac{dy}{du} \bigg|_{u=-3} \cdot \frac{du}{dx} \bigg|_{x=0} \]

\[ = (1 - 2u) \bigg|_{u=-3} (1) \bigg|_{x=0} \]

\[ = 1 - 2(-3) = 7 \]

(b) \( y = \left(\frac{u-1}{u+1}\right)^{1/2}; \; u = \sqrt{x-1} = (x-1)^{1/2} \)

\[ \frac{dy}{du} = \frac{1}{2} \left(\frac{u-1}{u+1}\right)^{-1/2} \left(\frac{u+1-u-1}{(u+1)^2}\right) \]

\[ = \frac{1}{(u-1)^{1/2}(u+1)^{3/2}} \]

\[ \frac{du}{dx} = \frac{1}{2} (x-1)^{-1/2} (1) = \frac{1}{2\sqrt{x-1}} \]

When \( x = \frac{34}{9}, \; u = \sqrt{\frac{25}{9}} = \frac{5}{3} \).

\[ \frac{dy}{dx} \bigg|_{x=\frac{34}{9}} = \frac{dy}{du} \bigg|_{u=\frac{5}{3}} \cdot \frac{du}{dx} \bigg|_{x=\frac{34}{9}} \]

\[ = \left(\frac{5}{3} - 1\right)^{1/2} \left(\frac{5}{3} + 1\right)^{3/2} \cdot \frac{1}{2\sqrt{\frac{53}{9}} - 1} \]

\[ = \frac{1}{\sqrt{\frac{3}{3}} \left(\frac{8}{3}\right)^{3/2}} \cdot \frac{1}{2\sqrt{\frac{53}{9}} - 1} \]

\[ = \left(\frac{3^{1/2}}{2}\right)\left(\frac{3^{3/2}}{2}\right) \left(\frac{3}{(2^{1/2})(8^{3/2})(2)(5)}\right) \]

\[ = \frac{27}{320} \]
26. (a) \[ f(x) = 6x^5 - 4x^3 + 5x^2 - 2x + \frac{1}{x} \]
\[ f'(x) = 30x^4 - 12x^2 + 10x - 2 - \frac{1}{x^2} \]
\[ f''(x) = 120x^3 - 24x + 10 + \frac{2}{x^3} \]

(b) \[ z = \frac{2}{1 + x^2} = 2(1 + x^2)^{-1} \]
\[ \frac{dz}{dx} = -2(1 + x^2)^{-2} \cdot 2x = -\frac{4x}{(1 + x^2)^2} \]
\[ \frac{d^2z}{dx^2} = \frac{8x(1 + x^2)(4) - 4x[2(1 + x^2)(2x)]}{(1 + x^2)^4} \]
\[ = -\frac{4(1 - 3x^2)}{(1 + x^2)^3} \]
\[ = \frac{4(3x^2 - 1)}{(1 + x^2)^3} \]

(c) \[ y = \left(3x^2 + 2\right)^4 \]
\[ \frac{dy}{dx} = 4 \left(3x^2 + 2\right)^3 \cdot 6x = 24x \left(3x^2 + 2\right)^3 \]
\[ \frac{d^2y}{dx^2} = 24x \left[ \left(3x^2 + 2\right)^2 \cdot 6x \cdot (3) \right] + \left(3x^2 + 2\right)^3 \cdot 24 \]
\[ = 24 \left(3x^2 + 2\right)^2 \left(21x^2 + 2\right) \]

28. (a) \[ 5x + 3y = 12, \quad 5 + 3 \frac{dy}{dx} = 0, \quad \text{or} \quad \frac{dy}{dx} = -\frac{5}{3} \]

(b) \[ (2x + 3y)^5 = x + 1, \]
\[ 592x + 3y^4 \left(2 + 3 \frac{dy}{dx}\right) = 1 \]
\[ 10(2x + 3y)^4 + 15(2x + 3y)^4 \frac{dy}{dx} = 1 \]
\[ \frac{dy}{dx} = \frac{1 - 10(2x + 3y)^4}{15(2x + 3y)^4} \]
30. (a) \[ xy^3 = 8 \]

\[ x \left( 3 y^2 \frac{dy}{dx} \right) + y^3 = 0 \]

or \[ \frac{dy}{dx} = -\frac{y^3}{3xy^2} = -\frac{y}{3x} \]

To find the slope of the tangent line at the point (1, 2), substitute \( x = 1 \) and \( y = 2 \) into the equation for \( \frac{dy}{dx} \) to get \( m = \frac{dy}{dx} = -\frac{2}{3} \).

(b) \[ x^2 \frac{dy}{dx} + y(2x) - 2 \left[ x \left( 3 y^2 \frac{dy}{dx} \right) + y^3 (1) \right] = 2 + 2 \frac{dy}{dx} \]

\[ x^2 \frac{dy}{dx} + 2xy - 6xy^2 \frac{dy}{dx} - 2y^3 = 2 + 2 \frac{dy}{dx} \]

To find the slope of the tangent line at (0,3), substitute \( x = 0 \) and \( y = 3 \) into the derivative equation and solve for \( \frac{dy}{dx} \) to get

\[ -2(3)^3 = 2 + 2 \frac{dy}{dx} \]

\[ -54 = 2 + 2 \frac{dy}{dx} \]

or the slope is \( m = \frac{dy}{dx} = -28 \).

32. \[ 4x^2 + y^2 = 1 \]

\[ 8x + 2y \frac{dy}{dx} = 0 \]

\[ \frac{dy}{dx} = -\frac{8x}{2y} = -\frac{4x}{y} \]

\[ \frac{d^2 y}{dx^2} = \frac{-4y + 4x \frac{dy}{dx}}{y^2} \]

\[ = \frac{-4y + 4x \left( -\frac{4x}{y} \right)}{y^2} \]

\[ = \frac{-4y^2 - 16x^2}{y^3} \]

34. (a) \( s(t) = -16t^2 + 160t = 0 \) when \( t = 0 \) and \( t = 10 \). The projectile leaves the ground at \( t = 0 \) and returns 10 seconds later.
Chapter 2. Differentiation: Basic Concepts

(b) \( \frac{ds}{dt} = -32t + 160, \) thus
\[ \frac{ds}{dt} = -160 \text{ ft/sec at } t = 10. \]

(c) \( \frac{ds}{dt} = 0 \) at \( t = 5 \) and the maximum height is \( s(5) = -16(25) + 160(5) = 400 \text{ ft}. \)

36. (a) \( s(t) = 2t^3 - 21t^2 + 60t - 25 \) for \( 1 \leq t \leq 6. \)
\( v(t) = 6(t^2 - 7t + 10) = 6(t - 2)(t - 5) \)
The positive roots are \( t = 2, t = 5. \) \( v(t) > 0 \) for \( 1 < t < 2, 5 < t < 6, \) so the object advances.
For \( 2 < t < 5, v(t) < 0 \) so the object retreats.
\( a(t) = 6(2t - 7) = 0 \) if \( t = \frac{7}{2}. \)
\[ a(t) > 0 \] for \( \frac{7}{2} < t < 6 \) so the object accelerates. For \( 1 < t < \frac{7}{2} \) it decelerates.

(b) \( s(1) = 2 - 21 + 60 - 25 = 16, \)
\( s(2) = 16 - 84 + 120 - 25 = 27, \)
\( s(5) = 250 - 21(25) + 300 - 25 = 0, \)
\( s(6) = 432 - 21(36) + 360 - 25 = 11 \)
\( \Delta s = (27 - 16) + (27 - 0) + (11 - 0) = 49 \)

38. (a) Since \( N(x) = 6x^3 + 500x + 8,000 \)
is the number of people using the system after \( x \) weeks, the rate at which use of the system is changing after \( x \) weeks is
\( N'(x) = 18x^2 + 500 \) people per week and the rate after 8 weeks is
\( N'(8) = 1,652 \) people per week.

(b) The actual increase in the use of the system during the 8th week is \( N(8) - N(7) = 1,514 \) people.

40. Since the population in \( t \) months will be \( P(t) = 3t + 5t^{3/2} + 6,000, \) the rate of change of the population will be \( P'(t) = 3 + \frac{15}{2}t^{1/2}, \) and the percentage rate of change 4 months from now will be
\[ 100 \frac{P'(4)}{P(4)} = 100 \left( \frac{18}{6,052} \right) \]
\[ = 0.30\% \] per month.

42. The gross domestic product \( t \) years after 2000 is \( N(t) = t^2 + 6t + 300 \) billion dollars.
The derivative is \( N'(t) = 2t + 6 \)
At the beginning of the second quarter of 2008, \( t = 8.25. \)
The change in \( t \) during this quarter is \( h = 0.25. \) Hence the percentage change in \( N \) is
\[ 100 \frac{N'(8.25)h}{N(8.25)} = 100 \left( \frac{[2(8.25) + 6](0.25)}{8.25^2 + 6(8.25) + 300} \right) \approx 1.347\% \]
44. \( C(t) = -170.36t^3 + 1,707.5t^2 + 1,998.4t + 4,404.8 \)

(a) \( C'(t) = -511.08t^2 + 3,415t + 1,998.4 \)

\( C'(t) \) represents the rate of change in the number of cases of AIDS at time \( t \) in units of reported cases per year.

(b) \( C'(0) = 1,998.4 \). The epidemic was spreading at the rate of approximately 1,998 cases per year in 1984.

(c) The percentage rate of change in 1984 \( (t = 0) \) was \( 100 \frac{C'(0)}{C(0)} = 100 \left( \frac{1,998.4}{4,404.8} \right) = 45.4\% \).

The percentage rate of change in 1990 \( (t = 6) \) was \( 100 \frac{C'(6)}{C(6)} = 100 \left( \frac{4,089.52}{41,067.44} \right) = 9.96\% \).

46. \( P(t) = 1.035t^3 + 103.5t^2 + 6,900t + 230,000 \)

(a) \( P'(t) = 3.105t^2 + 207t + 6,900 \). \( P'(t) \) represents the rate of change of the population, in bacteria per day, after \( t \) days.

(b) After 1 day the population is changing at \( P'(1) = 7,110.105 \) or about 7,110 bacteria per day. After 10 days the population is changing at \( P'(10) = 9,280.5 \) or about 9,281 bacteria per day.

(c) The initial bacterial population is \( P(0) = 230,000 \) bacteria. The population has doubled when \( P(t) = 2(230,000) = 460,000 \) or \( 1.035t^3 + 103.5t^2 + 6,900t - 230,000 = 0 \).

Using the solving features of a graphing calculator yields \( t \approx 23.3 \) days as the approximate time until the population doubles. At that time the rate of change is \( P'(23.3) = 13,409 \) bacteria per day.

48. By the approximation formula, \( \Delta y \approx \frac{dy}{dx} \Delta x \)

To find \( \frac{dy}{dx} \) differentiate the equation \( Q = x^3 + 2xy^2 + 2y^3 \) implicitly with respect to \( x \). Since \( Q \) is to be held constant in this analysis, \( \frac{dQ}{dx} = 0 \). Thus

\[ 0 = 3x^2 + 4xy \frac{dy}{dx} + 2y^2 + 6y^2 \frac{dy}{dx} \]

or \( \frac{dy}{dx} = -\frac{3x^2 + 2y^2}{4xy + 6y^2} \)

At \( x = 10 \) and \( y = 20 \)

\[ \frac{dy}{dx} = -\frac{3(10)^2 + 2(20)^2}{4(10)(20) + 6(20)^2} = -0.344 \]

Use the approximation formula with \( \frac{dy}{dx} = -0.344 \) and \( \Delta x = 0.5 \) to get \( \Delta y = -0.344(0.5) = -0.172 \) unit.

That is, to maintain the current level of output, input \( y \) should be decreased by approximately 0.172 unit to offset a 0.5 unit increase in input \( x \).
50. The population is
\[ p(t) = 10 - \frac{20}{(t + 1)^2} = 10 - 20(t + 1)^{-2} \]
and the carbon monoxide level is
\[ c(p) = 0.8\sqrt{p^2 + p + 139} \]
\[ = 0.8(p^2 + p + 139)^{1/2} \]
By the chain rule, the rate of change of the carbon monoxide level with respect to time is
\[ \frac{dc}{dt} = \frac{dc}{dp} \frac{dp}{dt} \]
\[ = 0.4(p^2 + p + 139)^{-1/2} (2p + 1) \left[ 40(t + 1)^{-3} \right] \]
\[ = \frac{0.4(2p + 1) 40}{\sqrt{p^2 + p + 139} (t + 1)^3} \]
At \( t = 1, \ p = p(1) = 10 - \frac{20}{4} = 5 \), \( c = c(5) = 0.8\sqrt{169} = 10.4 \).
The percentage rate of change is
\[ 100 \frac{dc}{dt} = 100 \frac{0.4(10 + 1) 40}{\sqrt{169}} \frac{1}{(1+1)^3} 10.4 \approx 16.27\% \text{ per year}. \]

52. \( V = x^3 \) and \( dV = 3x^2 \, dx \), \( \frac{dV}{V} = 3 \frac{x^2 \, dx}{x^3} = 3 \frac{dx}{x} = 0.06 \text{ or } 6\% \)

54. \( S(R) = 1.8 \left(10^5\right) R^2 \)
\( R = 1.2 \left(10^{-2}\right) \)
\( \Delta R = \pm 5 \left(10^{-4}\right) \)
\( \Delta S = S \left[ 1.2 \left(10^{-2}\right) \pm 5 \left(10^{-4}\right) \right] - S \left[ 1.2 \left(10^{-2}\right) \right] \)
\[ = S \left[ 1.2 \left(10^{-2}\right) \right] \left[ \pm 5 \left(10^{-4}\right) \right] \]
\( S' \left( R \right) = 3.6 \left(10^5\right) R \)
\( S' \left[ 1.2 \left(10^{-2}\right) \right] = \left[ 3.6 \left(10^5\right) \right] \left[ 1.2 \left(10^{-2}\right) \right] \)
\[ = 4.32 \left(10^3\right) \]
\( \Delta S = \left[ 4.3 \left(10^3\right) \right] \left[ \pm 5 \left(10^{-4}\right) \right] \)
\[ = \pm 2.15 \text{ cm/sec.} \]

56. \( V(t) = \left[ C_1 + C_2 P(t) \right] \left( \frac{3t^2}{T^2} - \frac{2t^3}{T^3} \right) \)
\[ \frac{dV}{dt} = \left[ C_1 + C_2 P(t) \right] \left( \frac{6t}{T^2} - \frac{6t^2}{T^3} \right) + C_2 \left( \frac{3t^2}{T^2} - \frac{2t^3}{T^3} \right) \frac{dP}{dt} \]
58. At \( t \) hours past noon, the truck is \( 70t \) km north of the intersection while the car is \( 105(t - 1) \) km east of the intersection. The distance between them is then
\[
D(t) = \sqrt{(70t)^2 + [105(t - 1)]^2} = \sqrt{4900t^2 + 11025(t - 1)^2}.
\]
The rate of change of the distance is
\[
D'(t) = \frac{9800t + 22050(t - 1)}{2\sqrt{4900t^2 + 11025(t - 1)^2}}.
\]
At 2 P.M., \( t = 2 \) and
\[
D'(2) = \frac{9800(2) + 22050(1)}{2\sqrt{4900(4) + 11025(1)^2}} = 119 \text{ km/hr}
\]
60. \( Q(t) = -t^3 + 9t^2 + 12t \), where 8 A.M. corresponds to \( t = 0 \).

(a) \( R(t) = Q'(t) = -3t^2 + 18t + 12 \)

(b) The rate at which the rate of production is changing is given by
\[ R'(t) = Q''(t) = -6t + 18 \]
At 9 A.M., \( t = 1 \), and \( R'(1) = 12 \) units per hour per hour.

(c) From 9:00 A.M. to 9:06 A.M. the change in time is 6 minutes or
\[ \Delta t = \frac{1}{10} \text{ hour.} \]
The change in the rate of production is approximated as
\[ \Delta R = R'(1)\Delta t = 12\left( \frac{1}{10} \right) = 1.2 \text{ units per hour.} \]

(d) The actual change, estimated in part (c), is
\[
R(1.1) - R(1) = Q'(1.1) - Q'(1)
= (-3(1.1)^2 + 18(1.1) + 12) - (-3 + 18 + 12)
= 1.17 \text{ units per hour.}
\]

62. (a) \( S(t) = 50\left(1 - \frac{t^2}{15}\right)^3 \)
\[ S(0) = 50 \text{ lbs.} \]

(b) \( S'(t) = 50(3)\left(1 - \frac{t^2}{15}\right)^2 \left[-\frac{2}{15}\right]t \)
\[ S'(1) = -150\left(1 - \frac{1}{15}\right)^2 \frac{2}{15} = -17.42 \text{ lbs/sec.} \]

(c) The bag will be empty when \( S(t) = 0 \) at \( t = \sqrt{15} = 3.873 \text{ sec.} \) The rate of leakage at that time is \( S'(\sqrt{15}) = 0 \).

64. \( C(q) = 0.1q^2 + 10q + 400 \), \( q(t) = t^2 + 50t \)

By the chain rule
\[ \frac{dC}{dt} = \frac{dC}{dq}\frac{dq}{dt} = (0.2q + 10)(2t + 50) \]
At \( t = 2 \), \( q = q(t) = 2^2 + 50(2) = 104 \) and
\[ \frac{dC}{dt} = [0.2(104) + 10][2(2) + 50] \]
\[ = 1,663.2 \text{ units per hour} \]

66. \( V = \frac{4}{3}\pi r^3 \)
\[ \Delta V = 0.08V \]
\[ V' = 4\pi r^2 \]
\[ \Delta V = 4\pi r^2 \Delta r \]
The percentage rate of change is
\[ \frac{100 \times 4\pi r^2 \Delta r}{\frac{4}{3}\pi r^3} = \frac{300 \Delta r}{r} \]
\[ 8 = (3) \left( \frac{100 \Delta r}{r} \right) \]
\[ \frac{100 \Delta r}{r} = \frac{8}{3} = 2.67 \]
The computations assumed a positive percentage rate of change of 8% but –8% could also be used. The percentage rate of change is then ±2.67%. 
68. Let $t$ be the time in hours and $s$ the distance between the car and the truck. Then
\[
s = \sqrt{(60t)^2 + (45t)^2} = \sqrt{3600t^2 + 2025t^2} = 75t
\]
and so $\frac{ds}{dt} = 75$ mph.

70. We have a right triangle with legs 8 and $x$, the distance from the buoy to the pier, and hypotenuse $y$, the length of the rope. Thus $y^2 = x^2 + 8^2$ and through implicit differentiation,
\[
2xx' = 2yy' \quad \text{or} \quad x' = \frac{yy'}{x}
\]
We know $y' = -2$ and at the moment in question, $x = 6$, so the rope is $y = 10$ ft long. Thus $x' = \frac{10(-2)}{6} = -\frac{10}{3}$. The buoy is approaching the pier at roughly 3.33 feet per minute.

72. Let $x$ be the distance between the woman and the building, and $s$ the length of the shadow. Since $h(t) = 150 - 16t^2$, the lantern will be 10 ft from the ground when $10 = 150 - 16t^2$ which leads to $t = \frac{1}{4}\sqrt{140}$ seconds.

When $h = 10$ and $x = 5t = \frac{5\sqrt{140}}{4}$ from similar right triangles we get
\[
\frac{x}{h-5} = \frac{x+s}{h}
\]
\[
\frac{5\sqrt{140}}{4(10-5)} = \frac{5\sqrt{140} + s}{10}
\]
or $s = \frac{5\sqrt{140}}{4}$
\[
hx = hx + hs - 5x - 5s
\]
\[
hs' + h's = 5\frac{dx}{dt} + 5s'
\]
\[
(h - 5)x' = 5\frac{dx}{dt} - h's
\]
\[
5s' = 5(5) + 32\left(\frac{1}{4}\right)\sqrt{140}\left(\frac{5}{4}\sqrt{140}\right)
\]
\[
s' = 5 + 2(140) = 285 \text{ ft/sec}
\]

74. The total manufacturing cost $C$ is a function of $q$ (where $q$ is the number of units produced) and $q$ is a function of $t$ (where $t$ is the number of hours during which the factory operates). Hence,

(a) $\frac{dC}{dq}$ = the rate of change of cost with respect to the number of units produced in dollars/unit.

(b) $\frac{dq}{dt}$ = the rate of change of units produced with respect to time in units/hour.

(c) $\frac{dC}{dq} \frac{dq}{dt}$ = the rate of change of cost WRT time in dollars/hour = the rate of change of cost WRT time.
76. $y = 4x^2$ and $P(2, 0)$

Note that $P$ is not on the graph of the curve (its coordinates do not satisfy the equation of the curve).

$y' = 8x$

Let $x_t$ be the abscissa of the point of tangency. The slope is $m = 8x_t$

The point $(x_t, y_t)$ lies on the curve through $(2, 0)$ so its slope is

$$\frac{y_t - 0}{x_t - 2} = 8x_t \quad \text{or} \quad y_t = 8x_t^2 = 16x_t$$

The point of contact (tangency) is both on the curve and on the tangent line. Thus

$$4x_t^2 = 8x_t^2 - 16x_t \quad \text{or} \quad 4x_t(x_t - 4) = 0.$$  

This is satisfied for $x_t = x_1 = 0$ as well as $x_t = x_2 = 4$.

The two points of contact have coordinates $(0, 0)$ and $(4, 64)$.

78. \[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1
\]

\[
\frac{2x}{a^2} - \frac{2y \frac{dy}{dx}}{b^2} = 0
\]

\[
\frac{dy}{dx} = \frac{b^2 x}{a^2 y}
\]

Thus the slope at $(x_0, y_0)$ is $m = \frac{b^2 x_0}{a^2 y_0}$ and the equation of the line becomes

\[
y - y_0 = \frac{b^2 x_0}{a^2 y_0} (x - x_0)
\]

\[
y_0 y - y_0 \frac{y_0^2}{b^2} = \frac{x_0 x}{a^2} - \frac{x_0^2}{a^2}
\]

\[
x_0 x - y_0 y = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1
\]

because the point $(x_0, y_0)$ is on the curve.
80. \[ f(x) = \frac{2x + 3}{1 - 3x} \]
\[ f'(x) = \frac{(1 - 3x)2 - (2x + 3)(-3)}{(1 - 3x)^2} \]
\[ = \frac{11}{(1 - 3x)^2} \]

It is clear from the graph and the expression for \( f'(x) \) that \( f'(x) \) is never 0.

82. \[ s(t) = t^{5/2}(0.73t^2 - 3.1t + 2.7) \]
\[ = 0.73t^{9/2} - 3.1t^{7/2} + 2.7t^{5/2} \]

(a) \[ v(t) = 3.285t^{7/2} - 10.85t^{5/2} + 6.75t^{3/2} \]
\[ a(t) = 11.4975t^{5/2} - 27.125t^{3/2} + 10.125t^{1/2} \]

(b) \( v(t) = 0 \) at \( t = 0 \) and \( t = 0.831 \). The corresponding positions at these times are \( s(0) = 0 \) and \( s(0.831) = 0.395 \).

(c) The smallest value of \( a(t) \) occurs at approximately \( t = 1.278 \). At this time \( s(1.278) = -0.128 \) and \( v(1.278) = -2.53 \).